

# Modern Algebra I

## *Lecture 5*

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*Today, we will continue the section*

## Section I.7: Categories: Products, Coproducts, and Free Objects

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- a morphism  $(\alpha, \beta)$  of  $\mathcal{D}$  is an equivalence if and only if  $\alpha$  and  $\beta$  are both equivalence of  $\mathcal{C}$ .

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This completes this part of proof.

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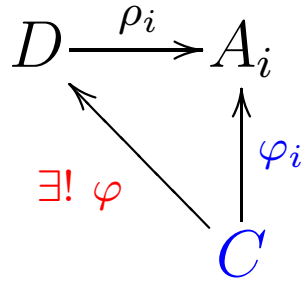
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$$\begin{array}{ccc} D & \xrightarrow{\rho_i} & A_i \\ & & \uparrow \varphi_i \\ & & C \end{array}$$

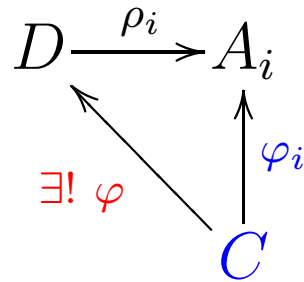
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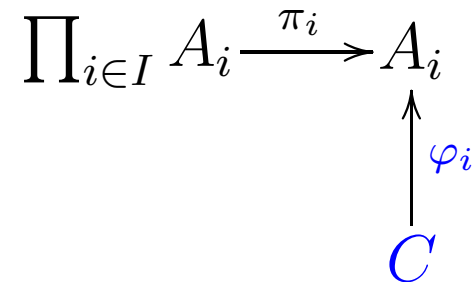


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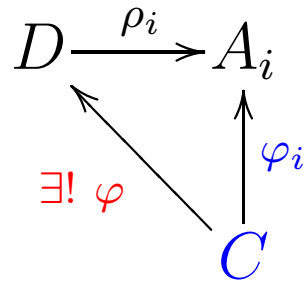


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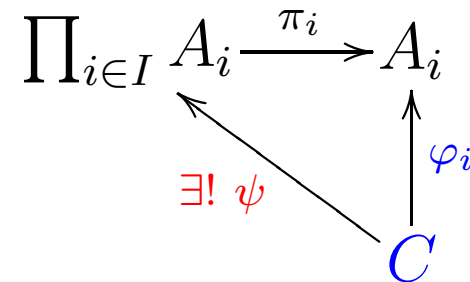


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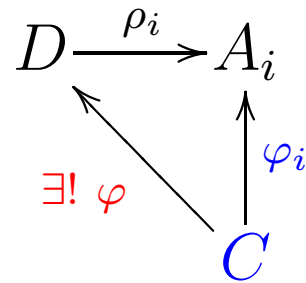


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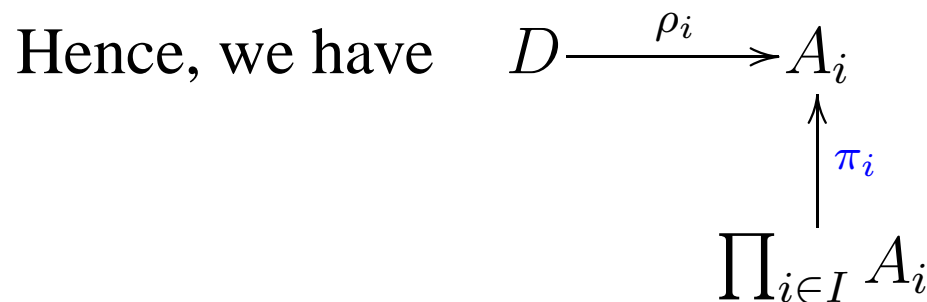
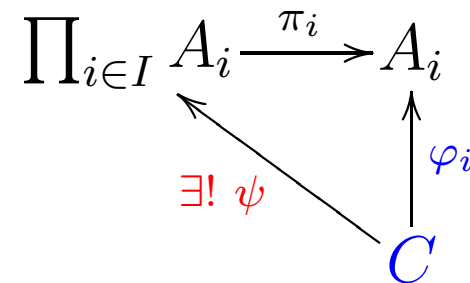


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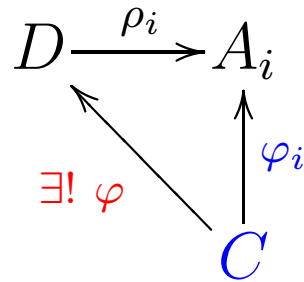


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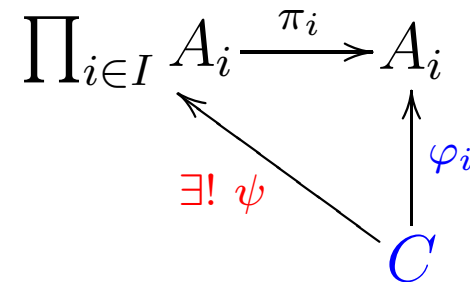


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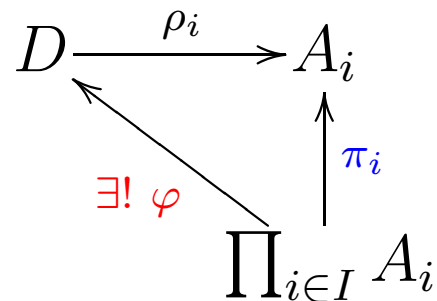
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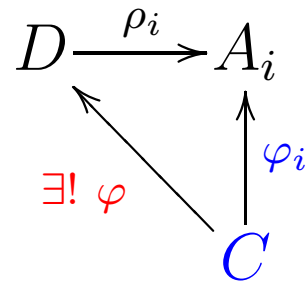


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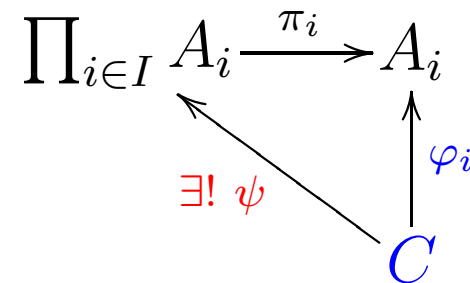


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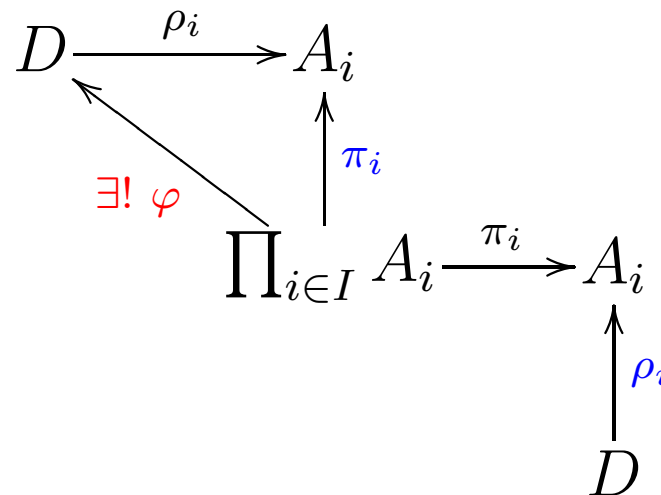
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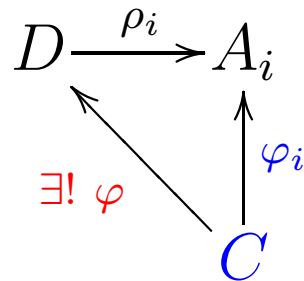


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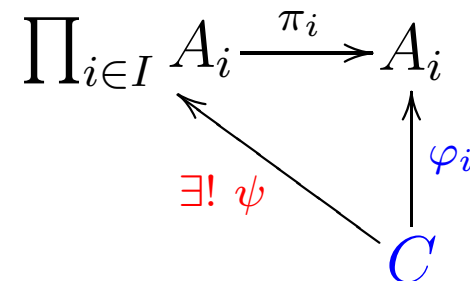


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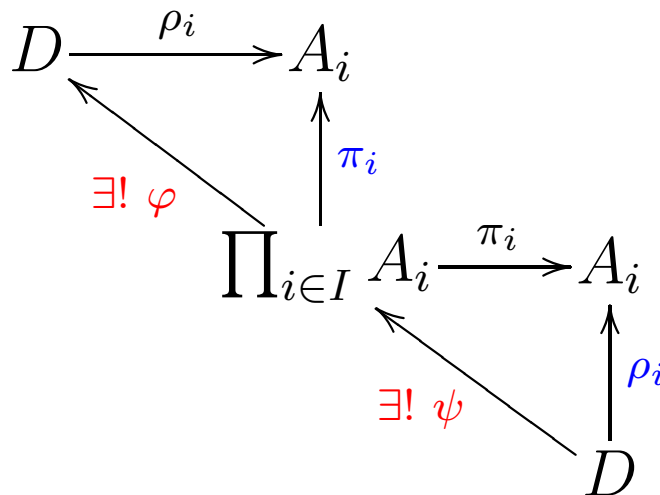
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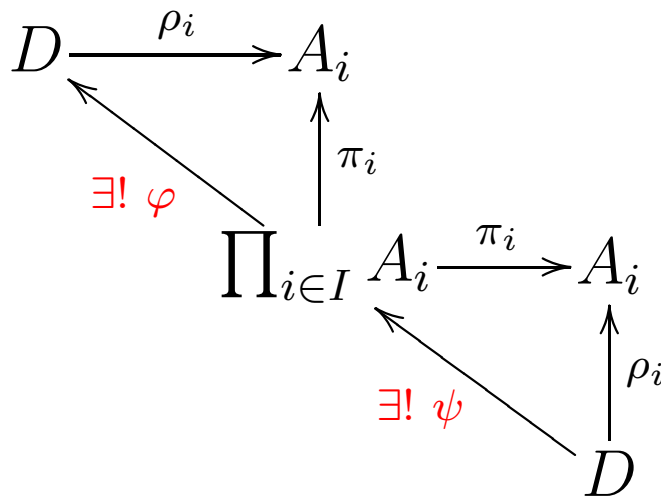


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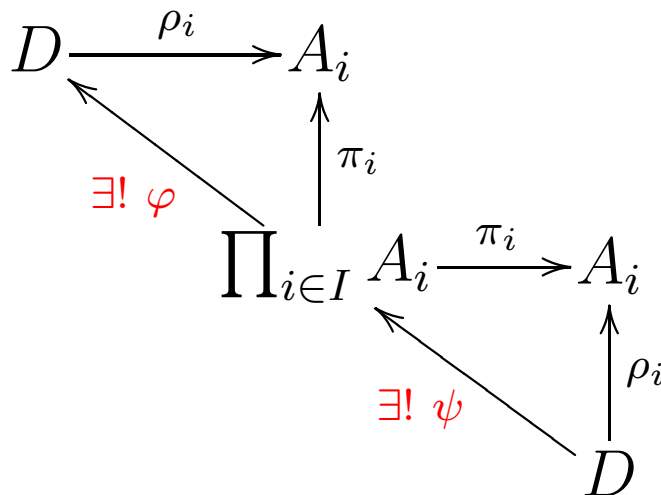
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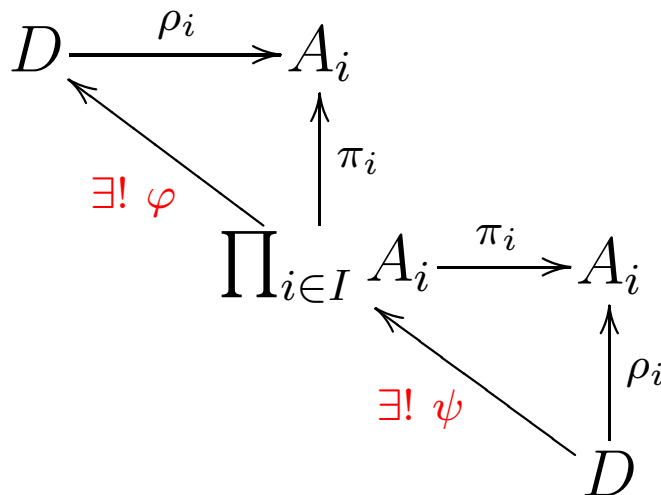
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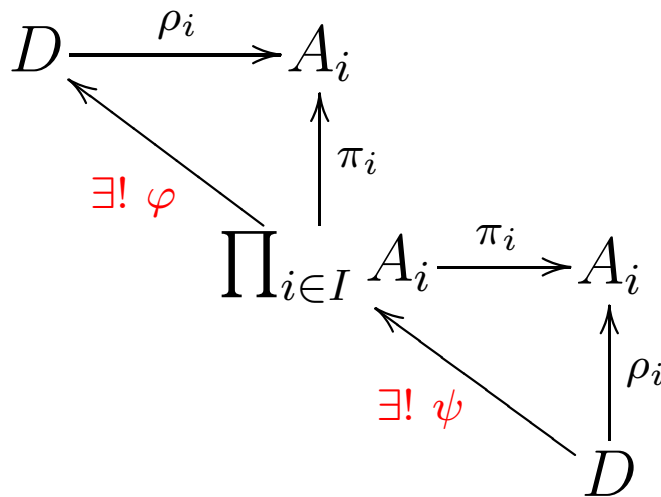
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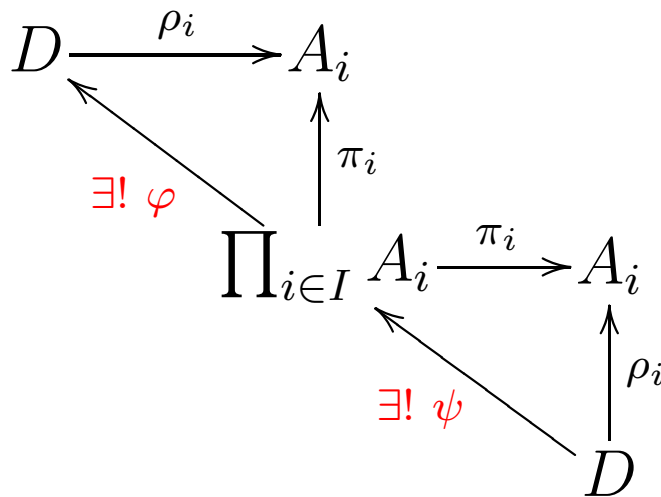
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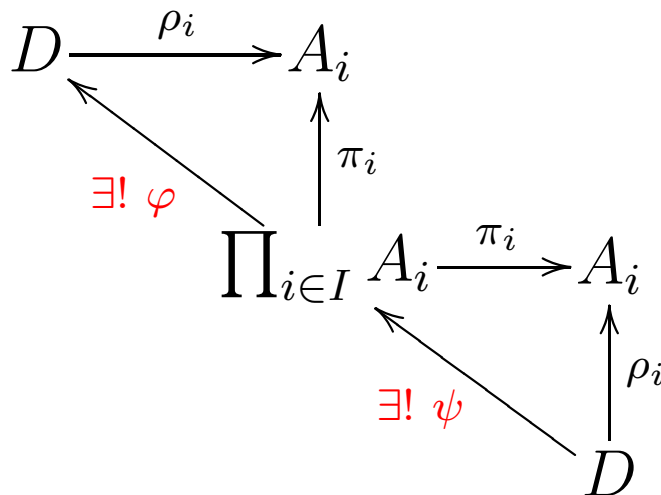
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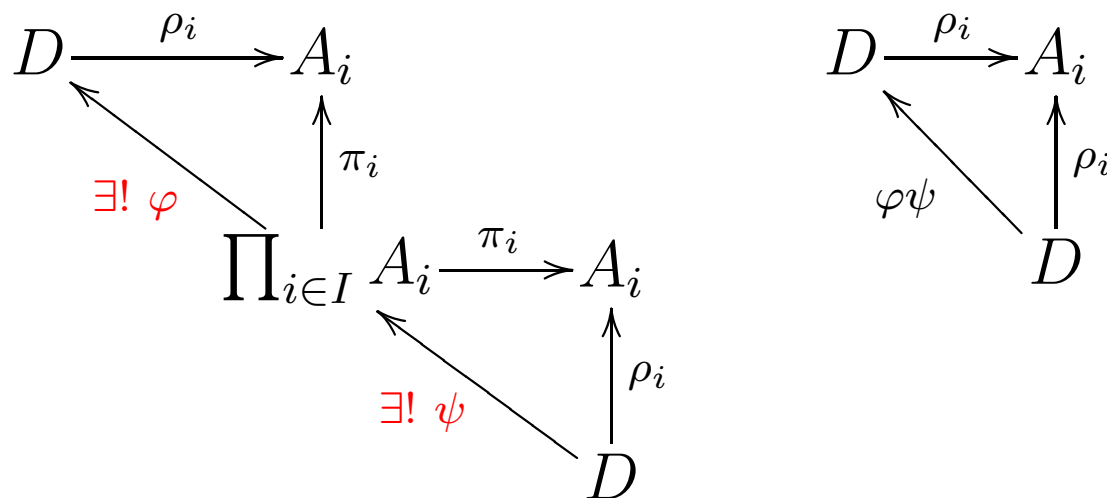
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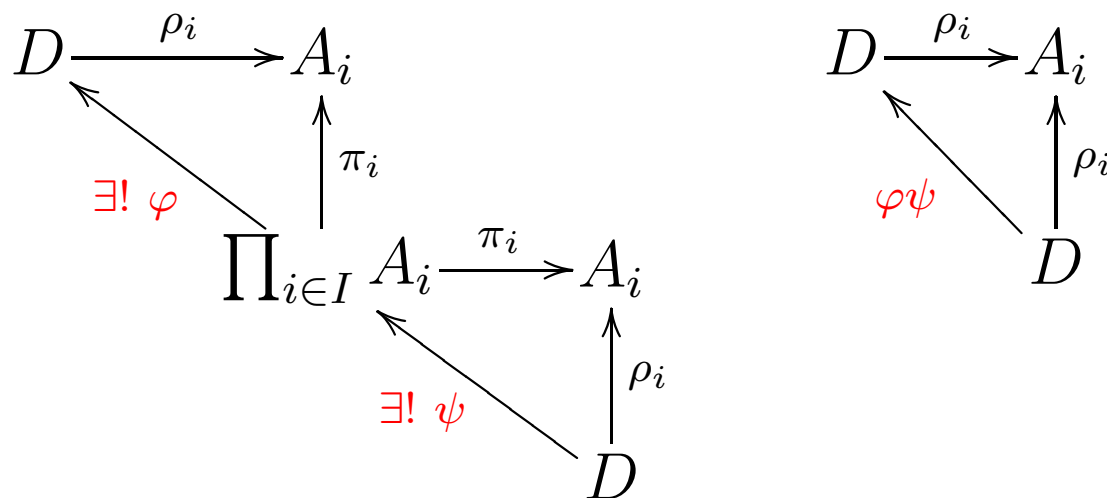
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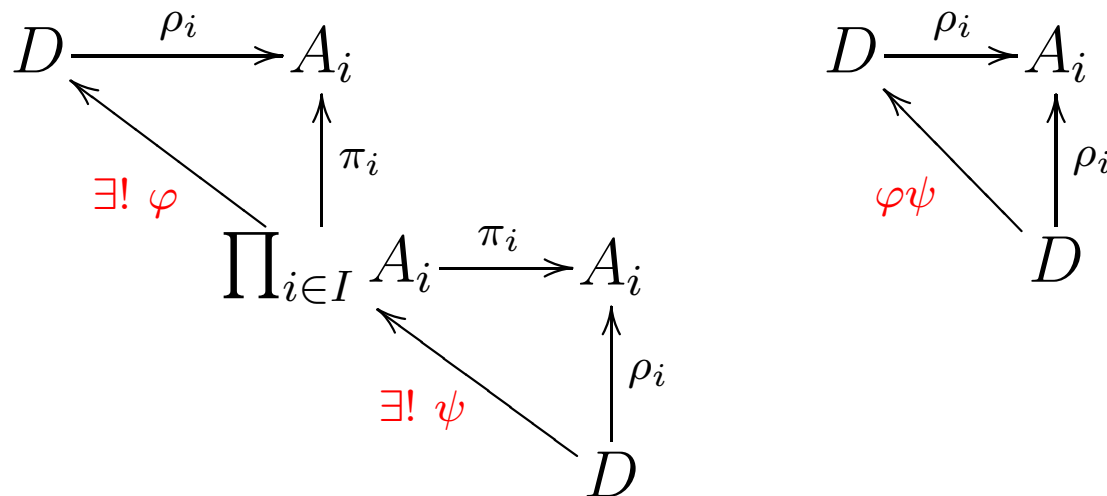
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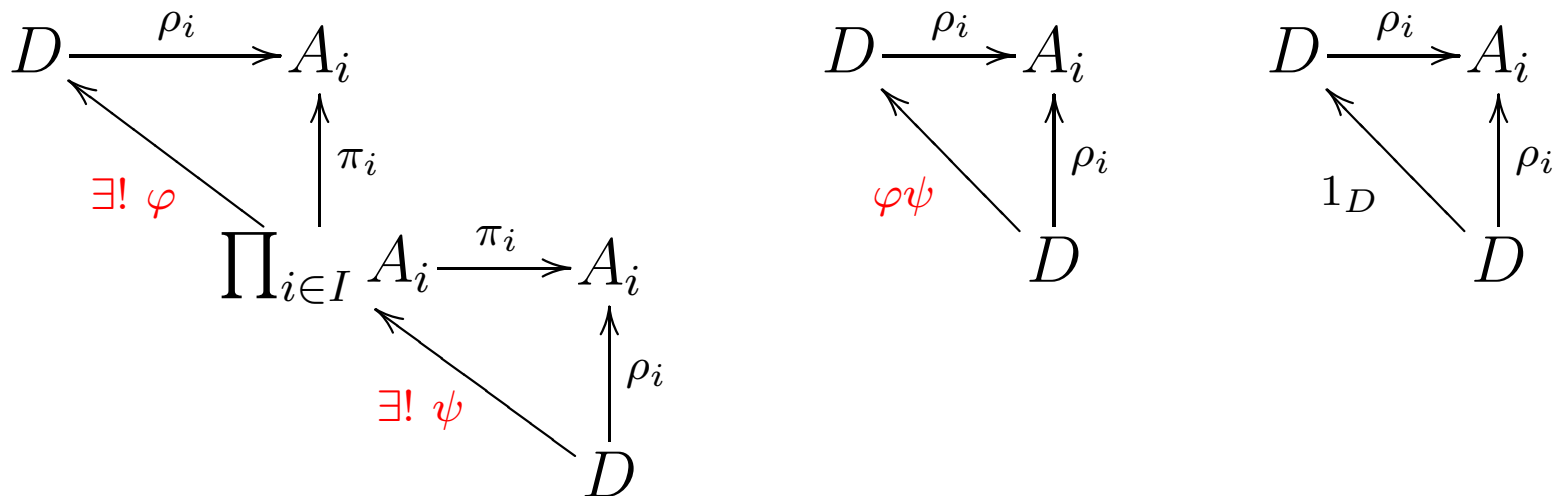
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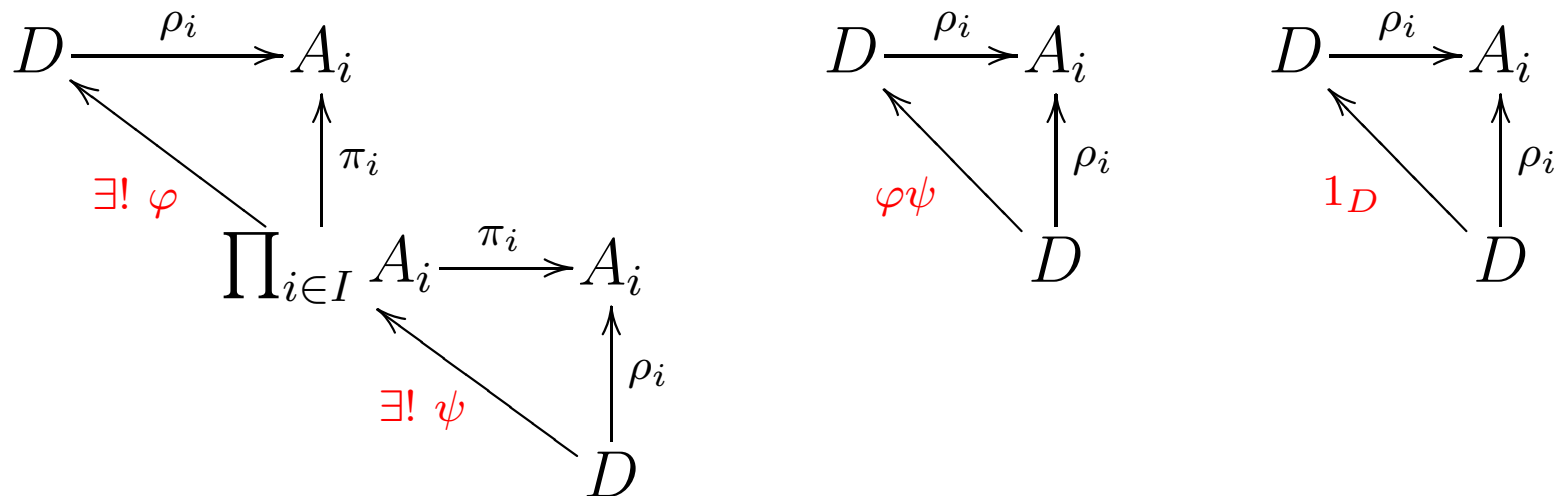
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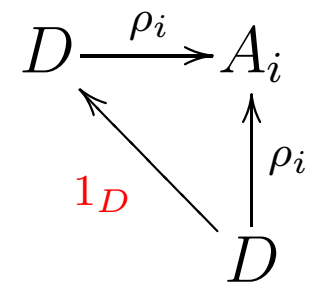
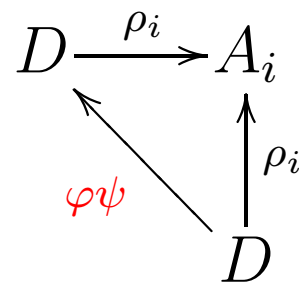
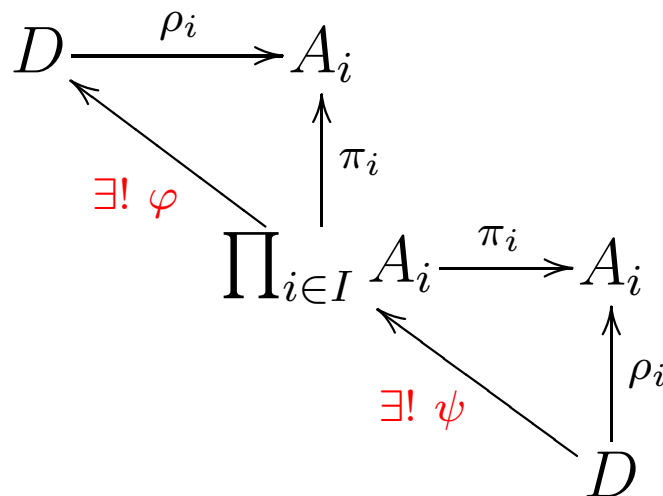
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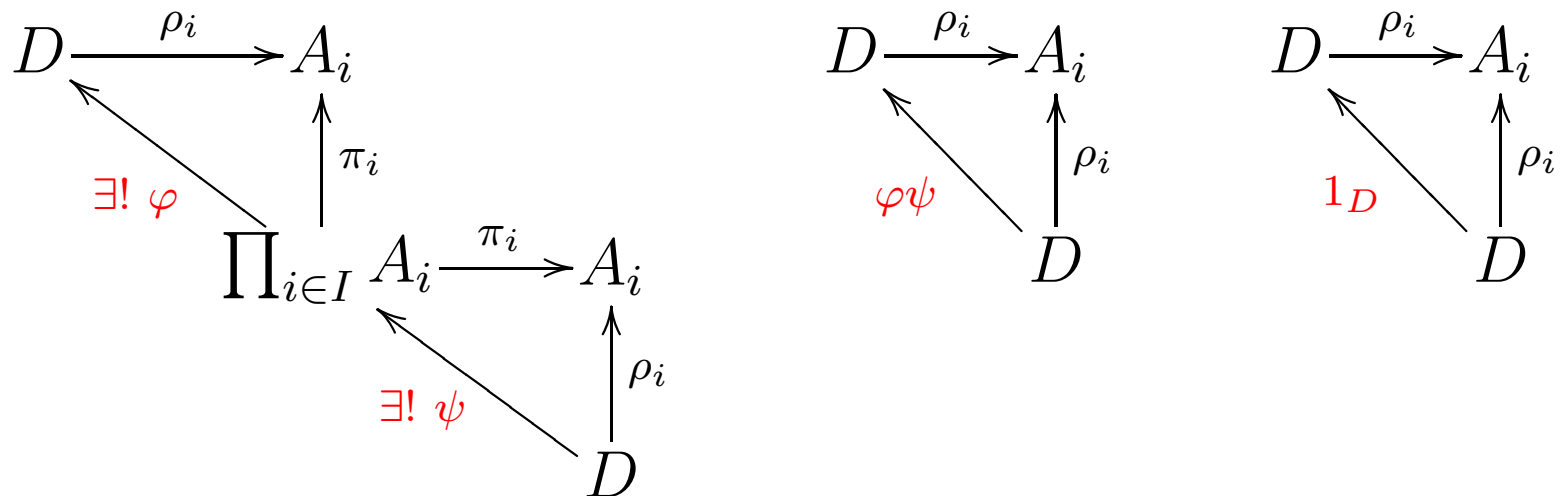
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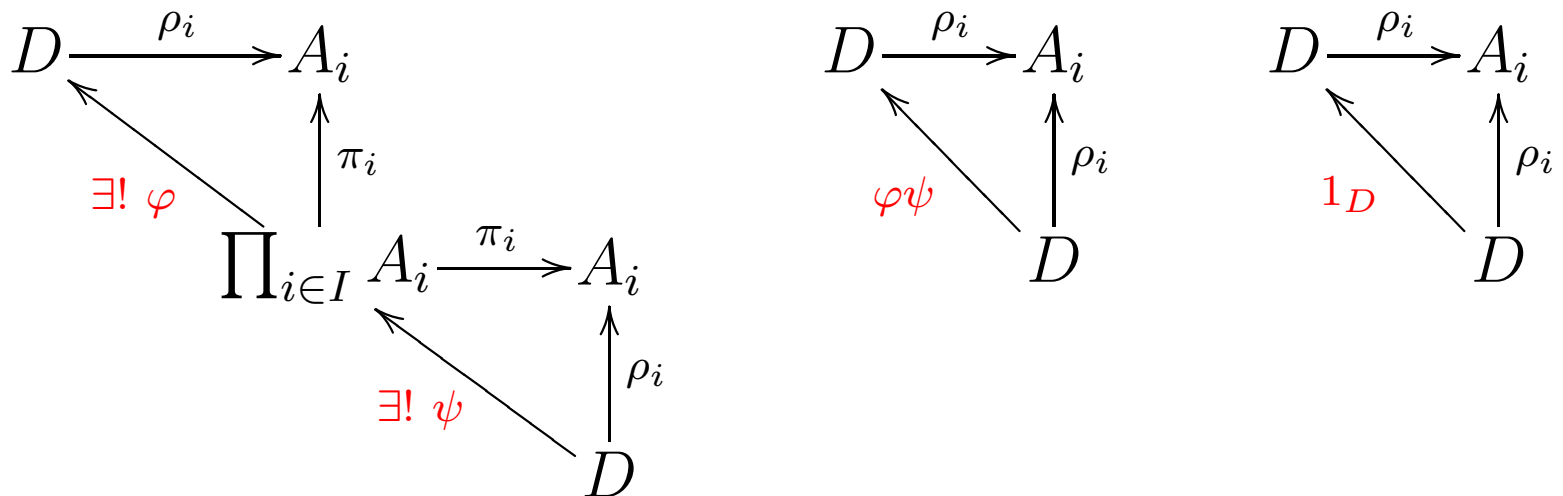
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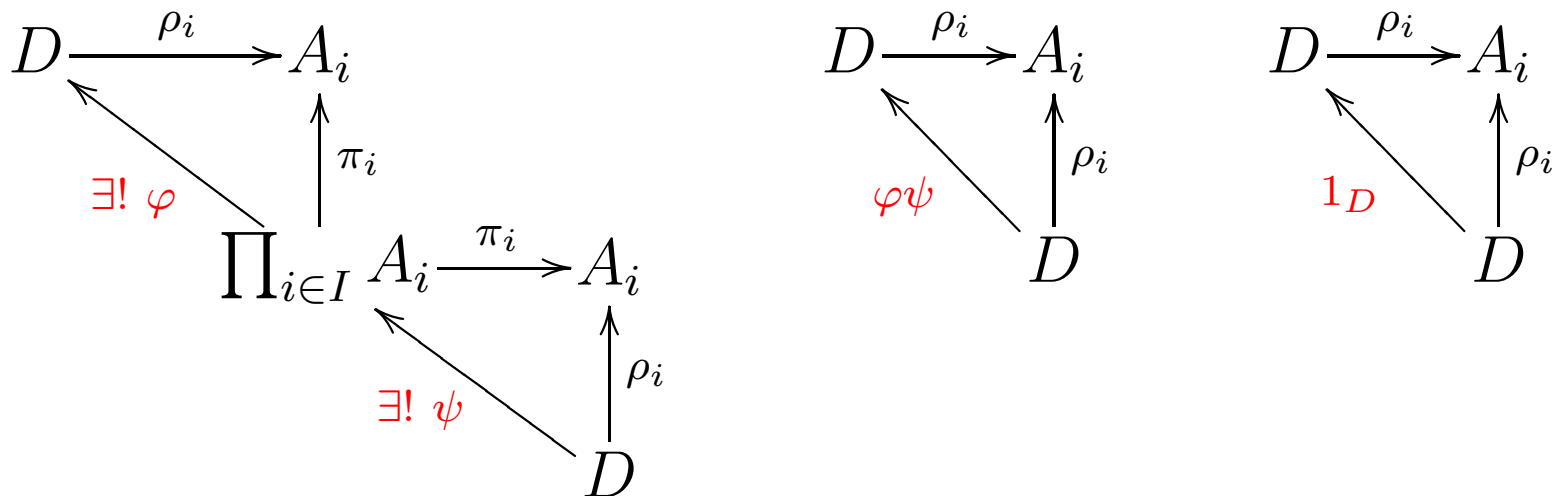
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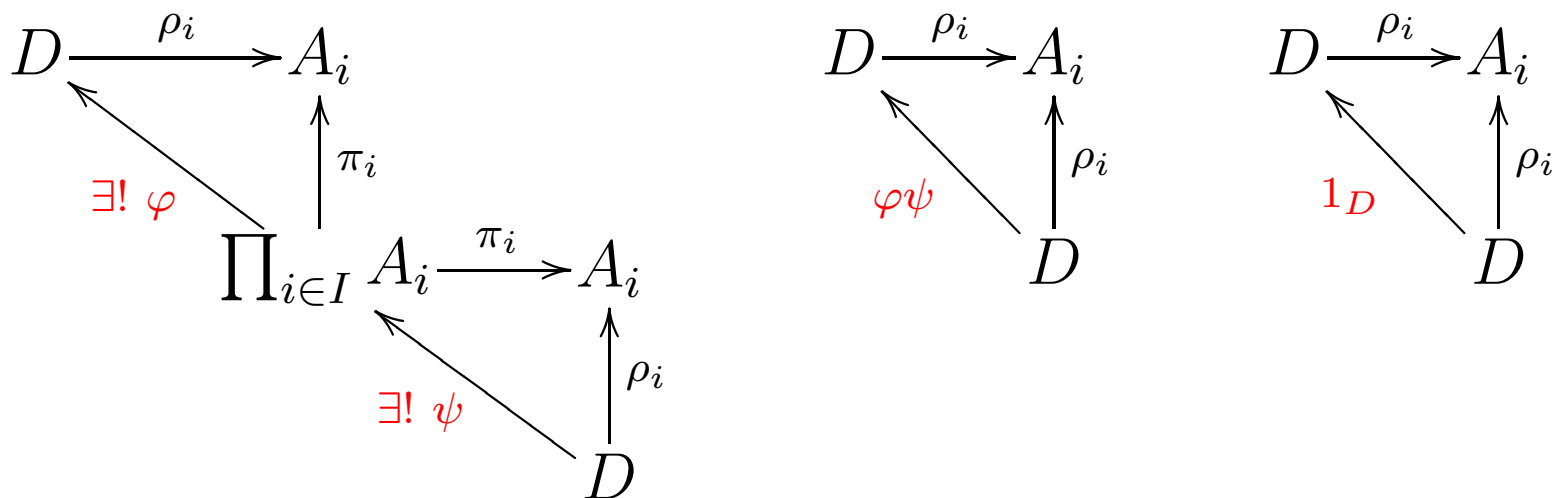
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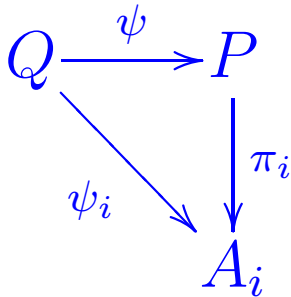
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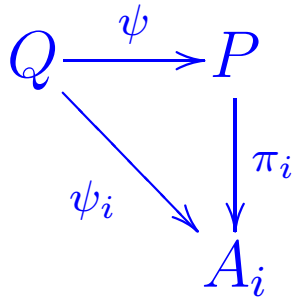


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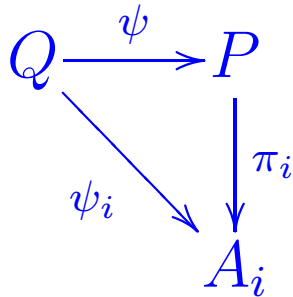
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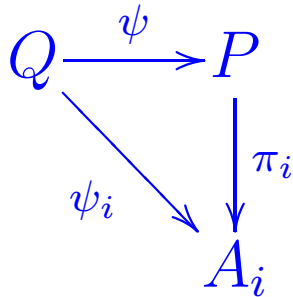


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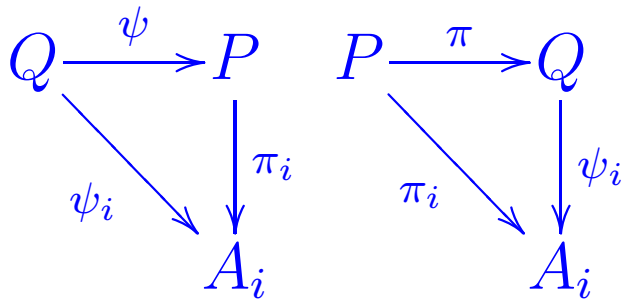


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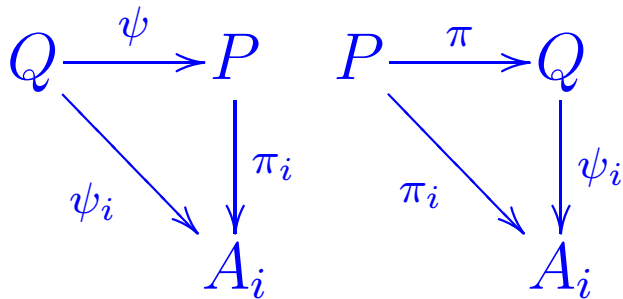


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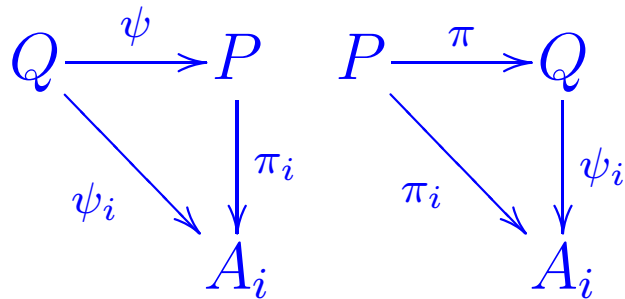


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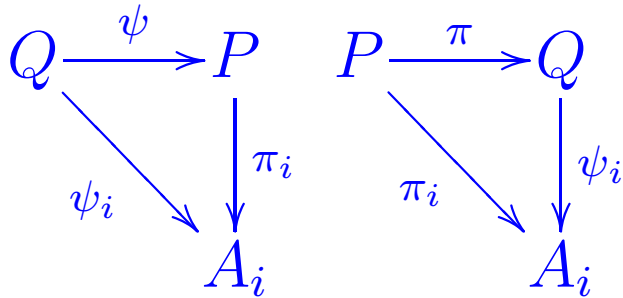


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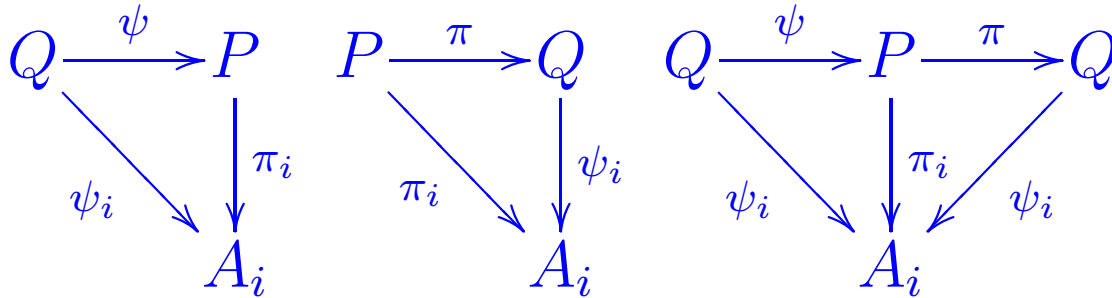


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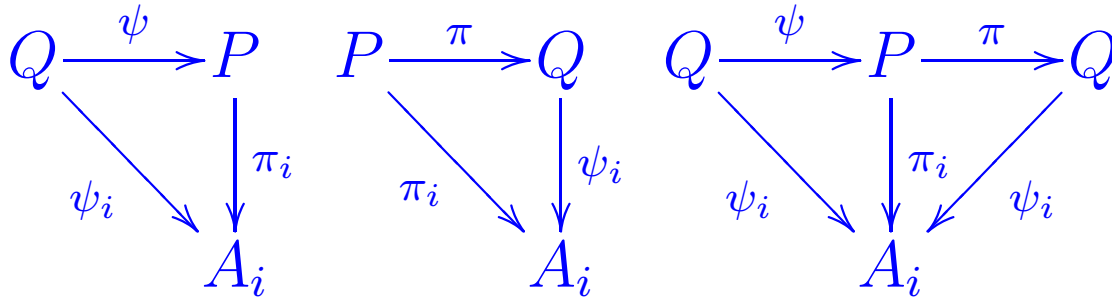


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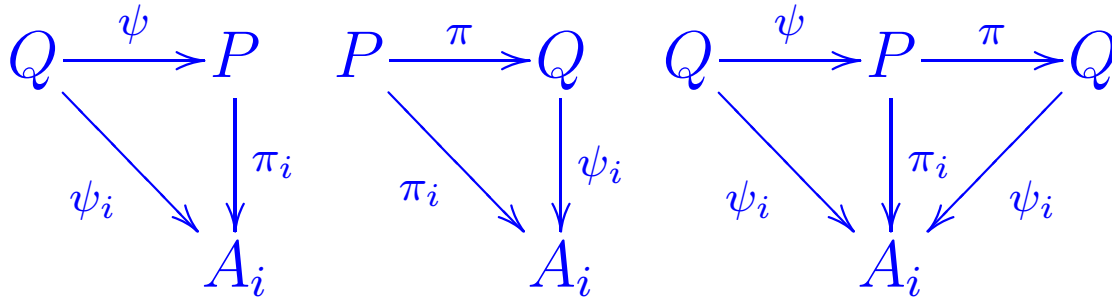


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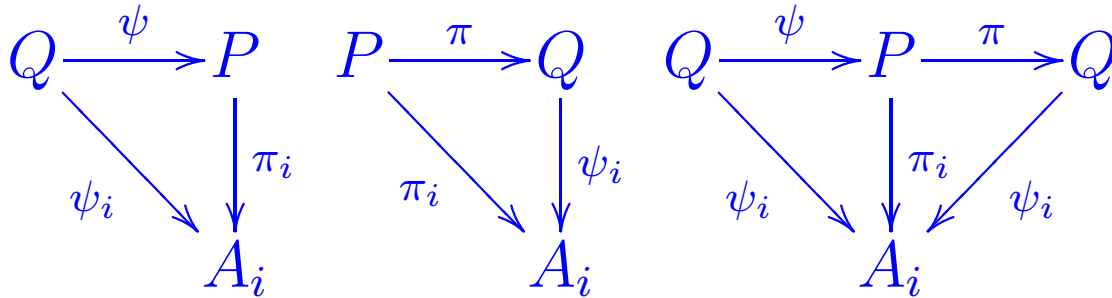


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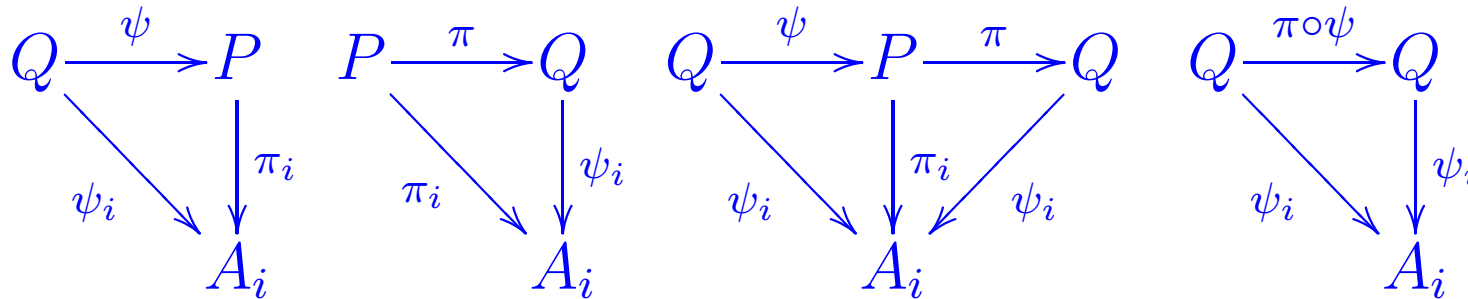


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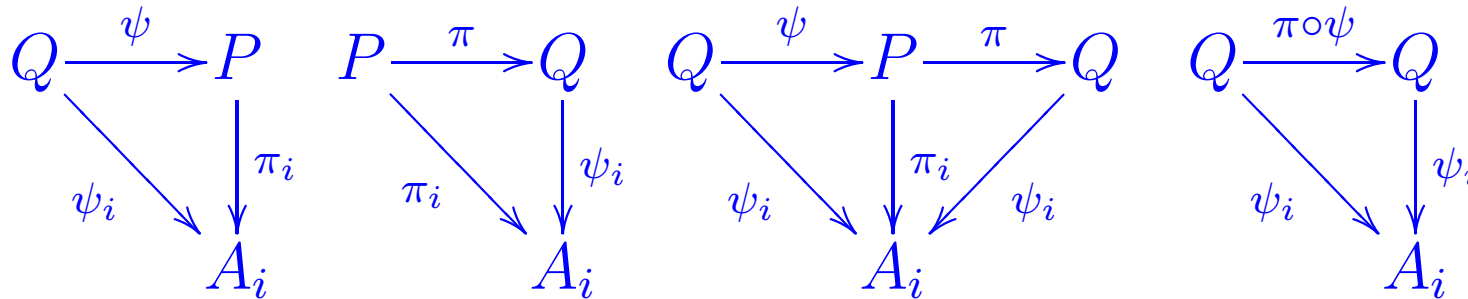


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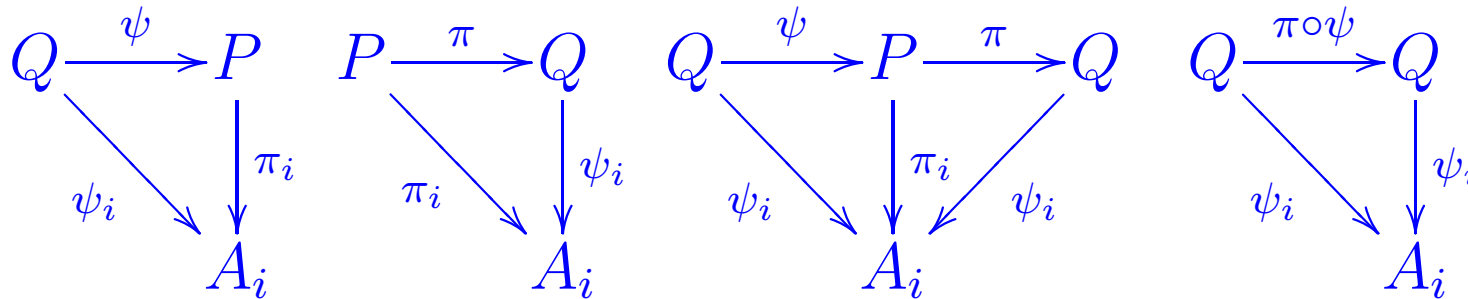


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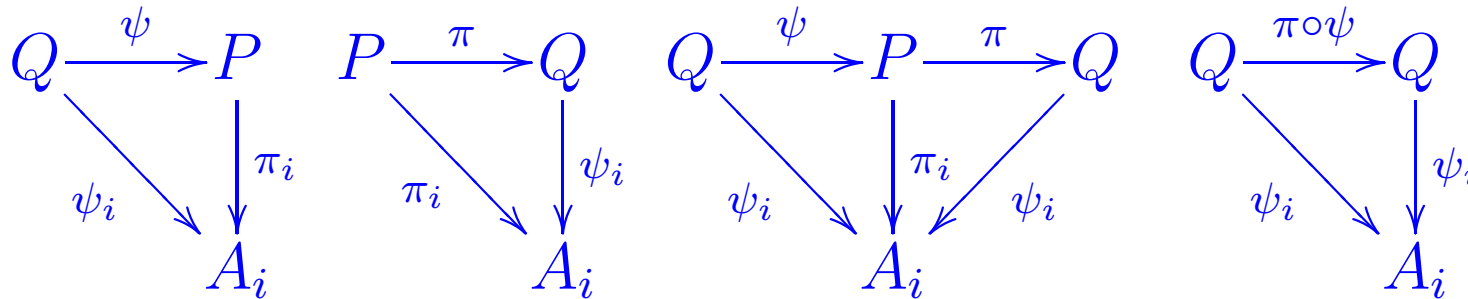


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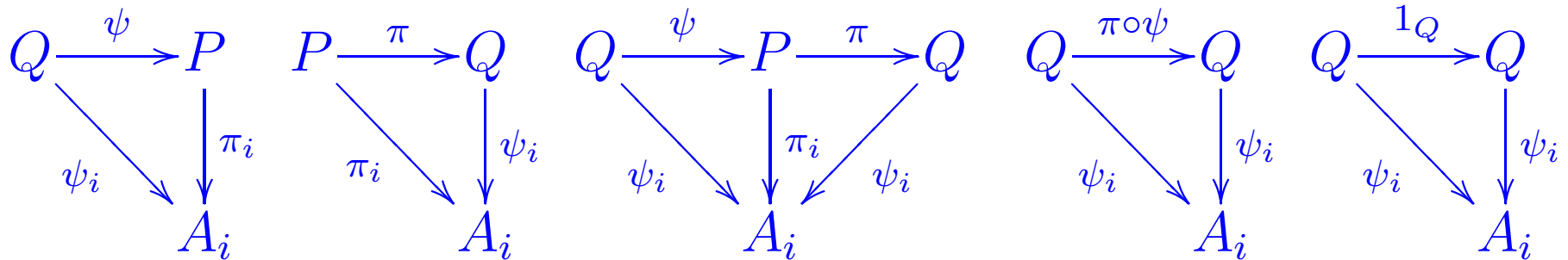


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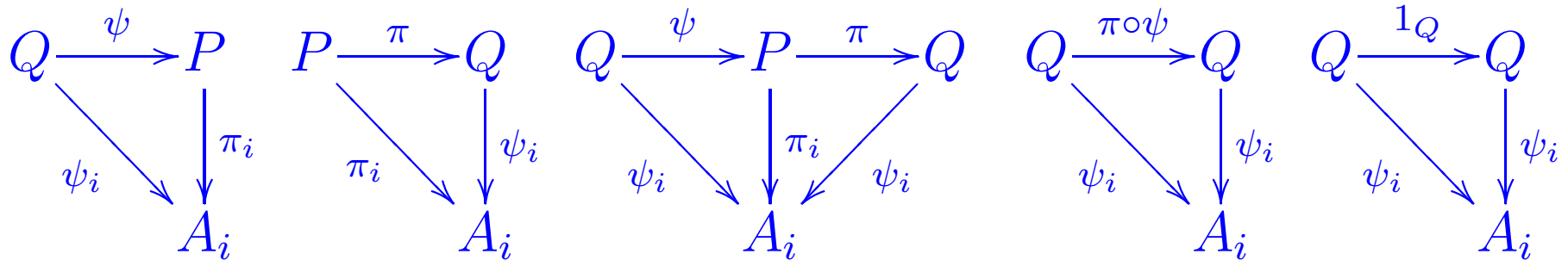


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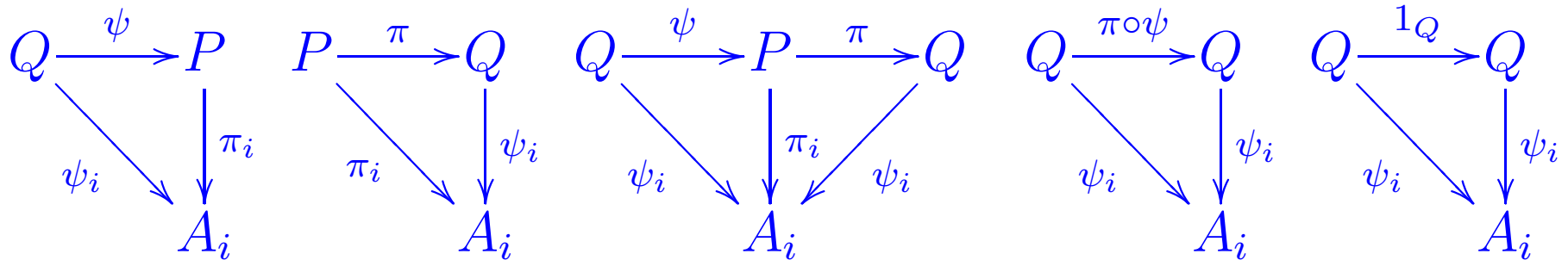


Combine the above two commutative diagrams and we have the third commutative diagram above for all  $i \in I$ . In other words, we have the fourth commutative diagram above for all  $i \in I$ . On the other hand, it is clear that we have the fifth commutative diagram above. By the uniqueness of the morphism  $Q \rightarrow Q$  that makes the diagram commute,

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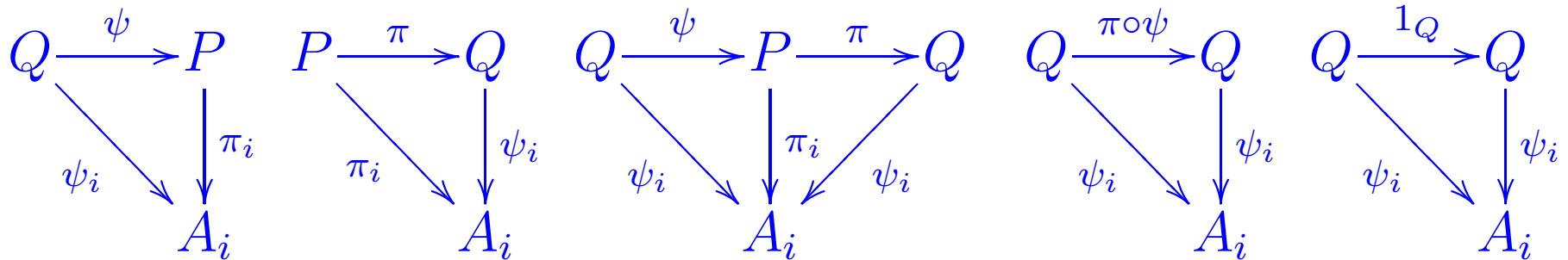


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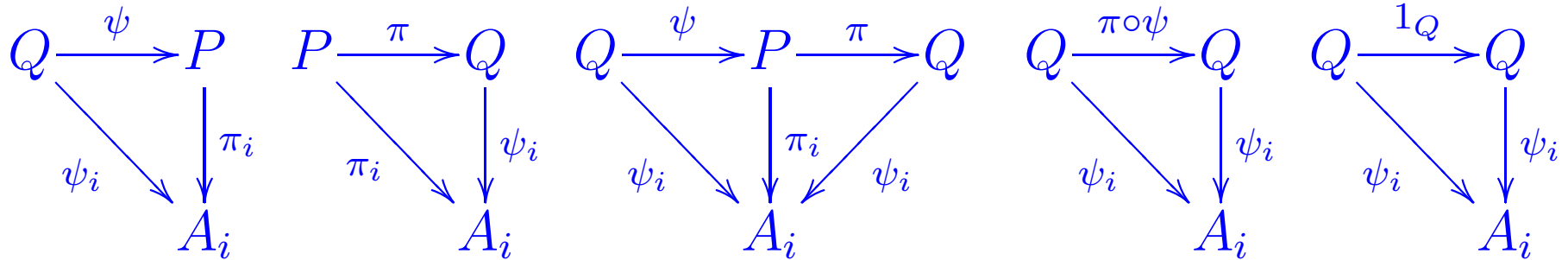


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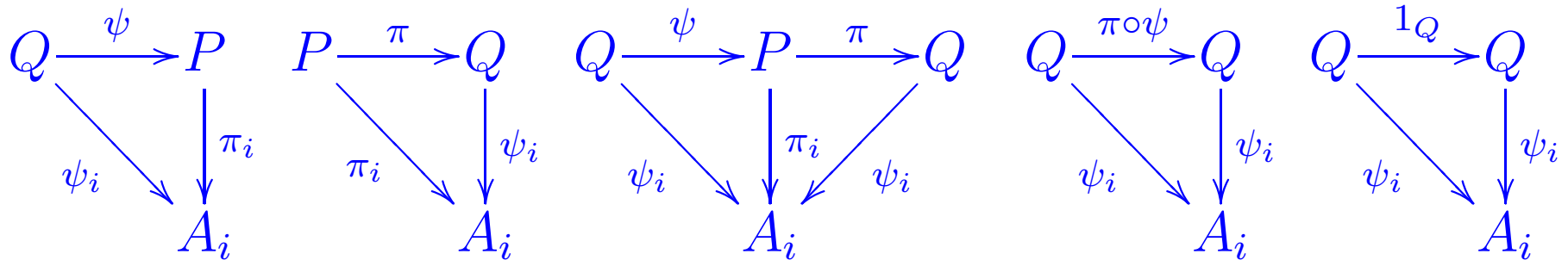


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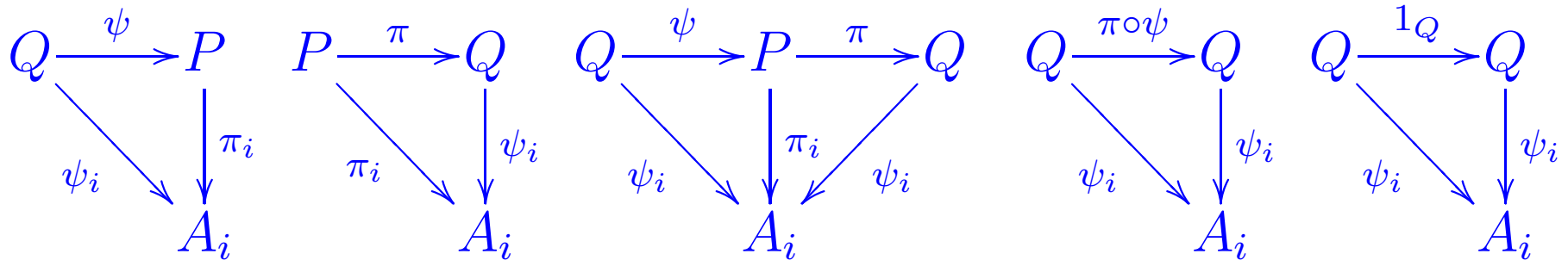


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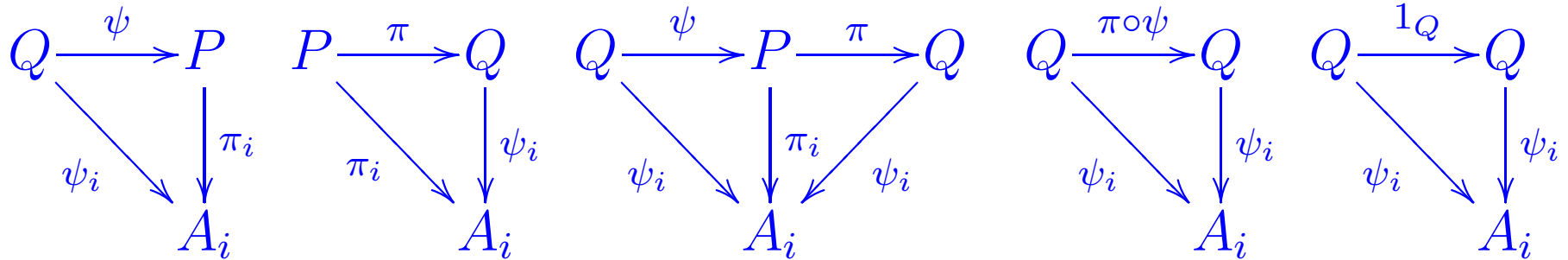


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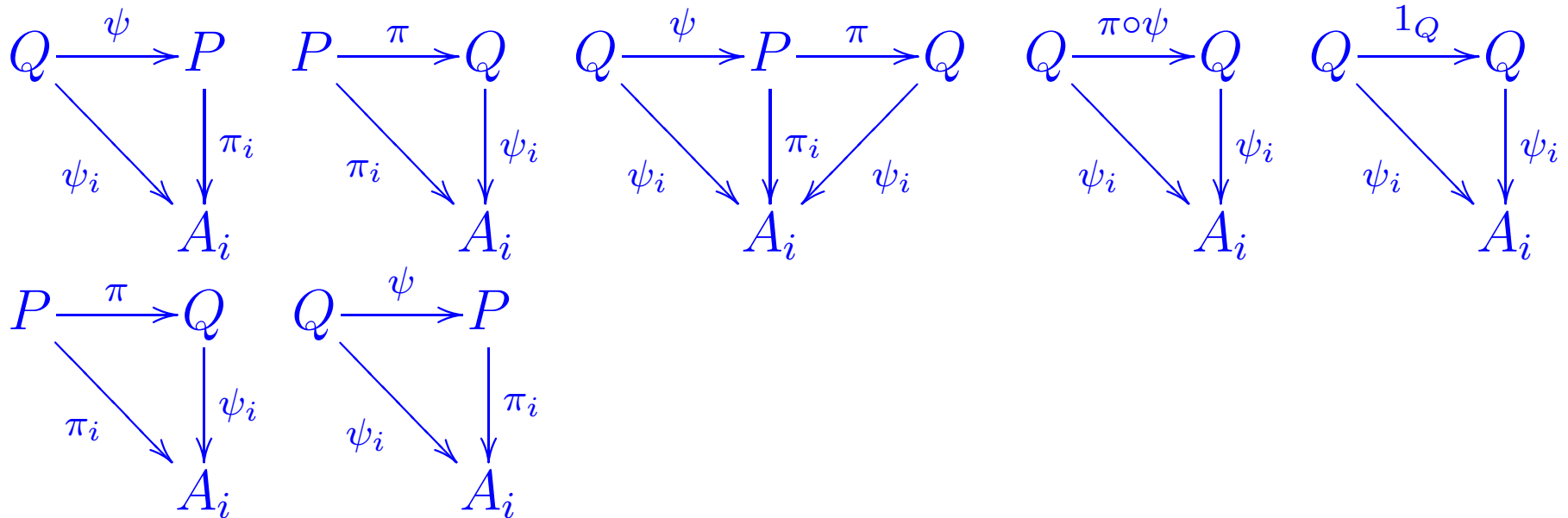


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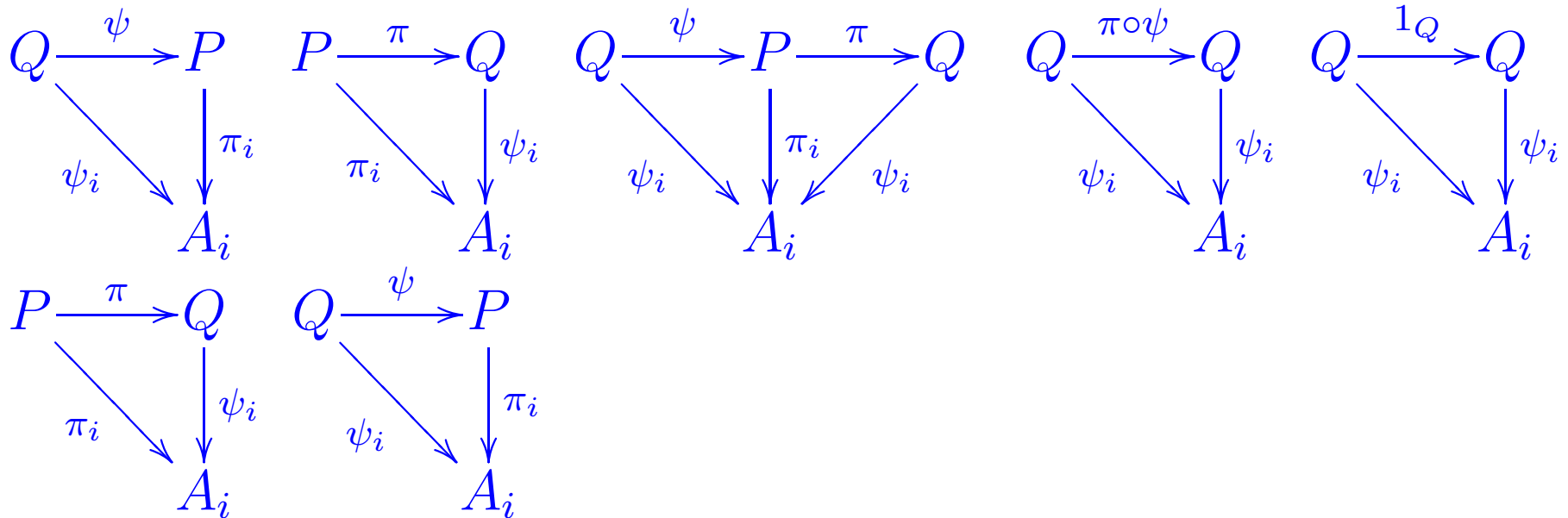


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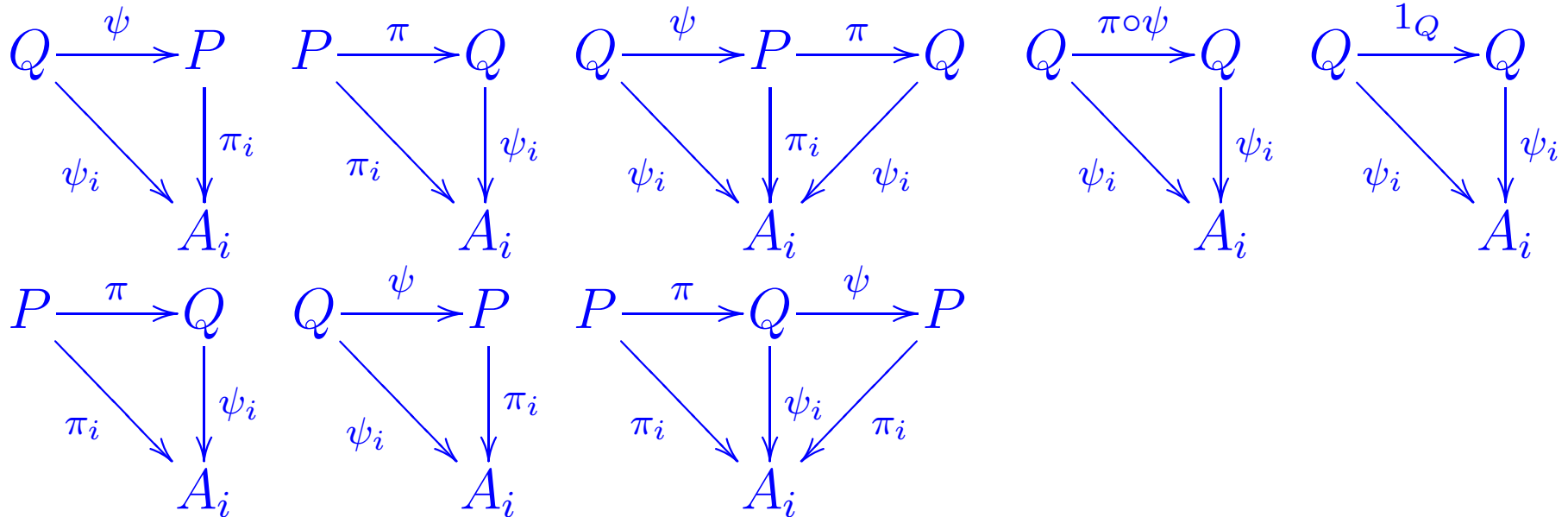


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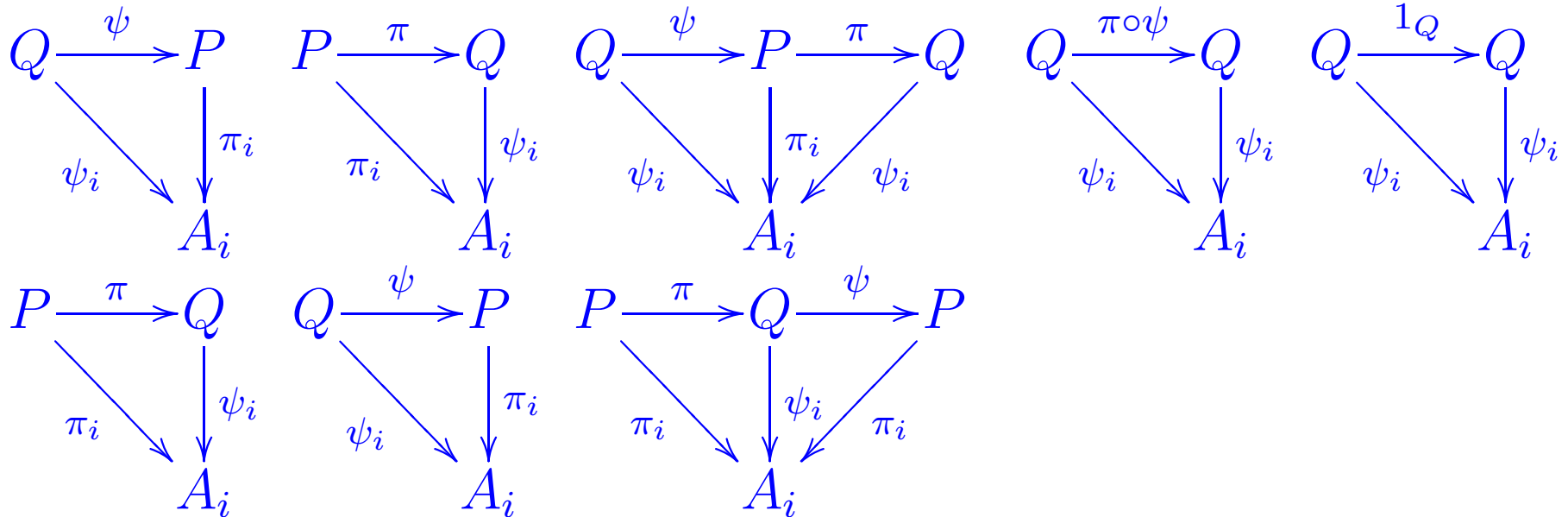


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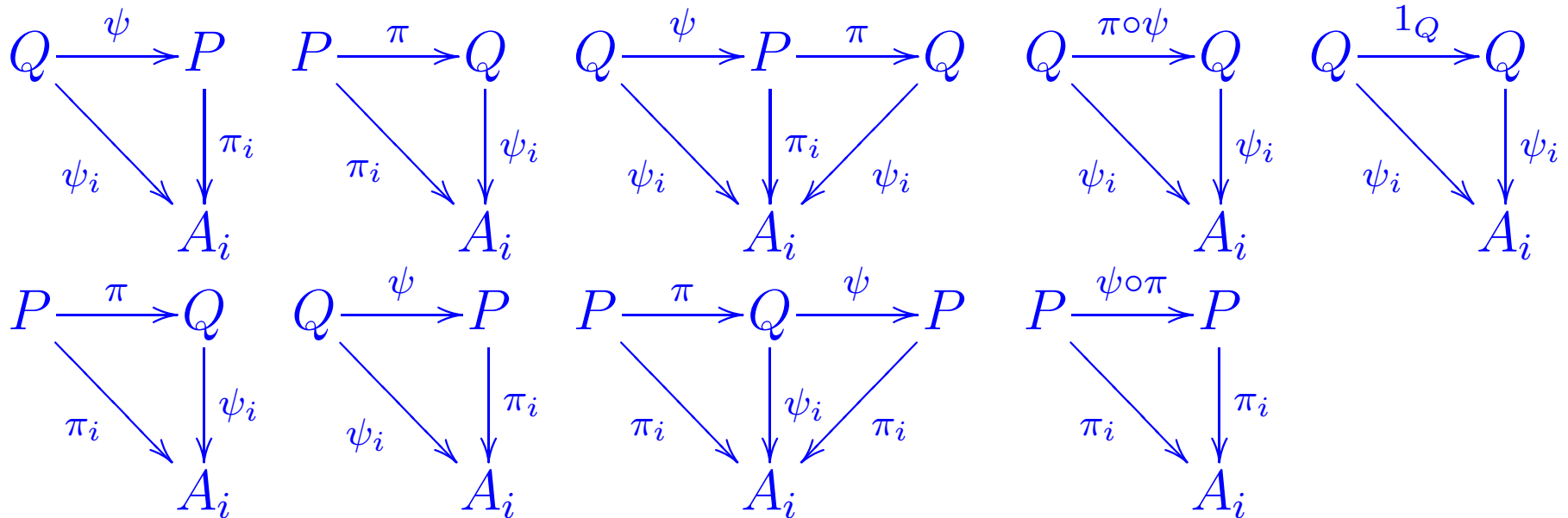


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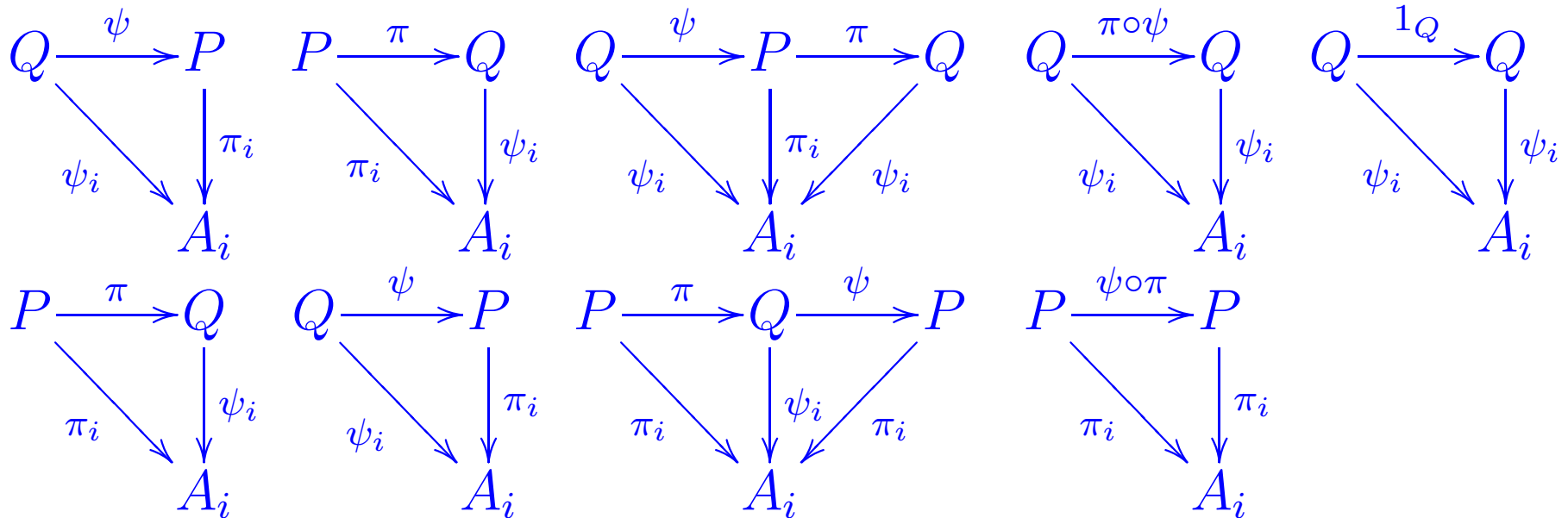


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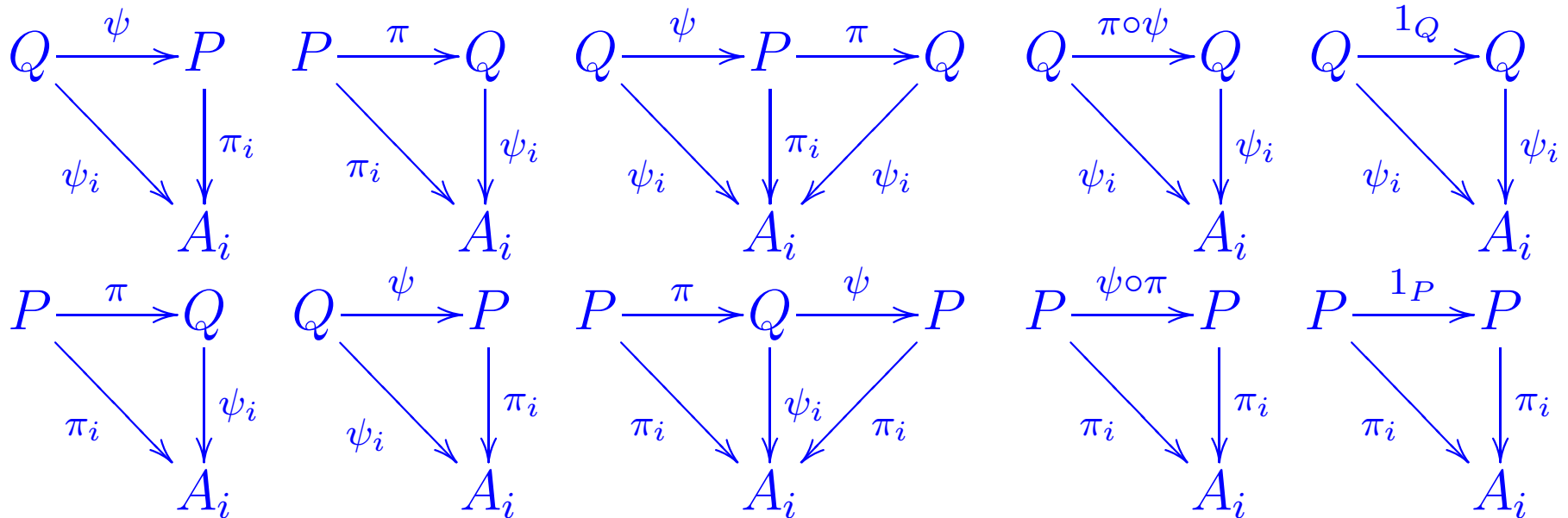


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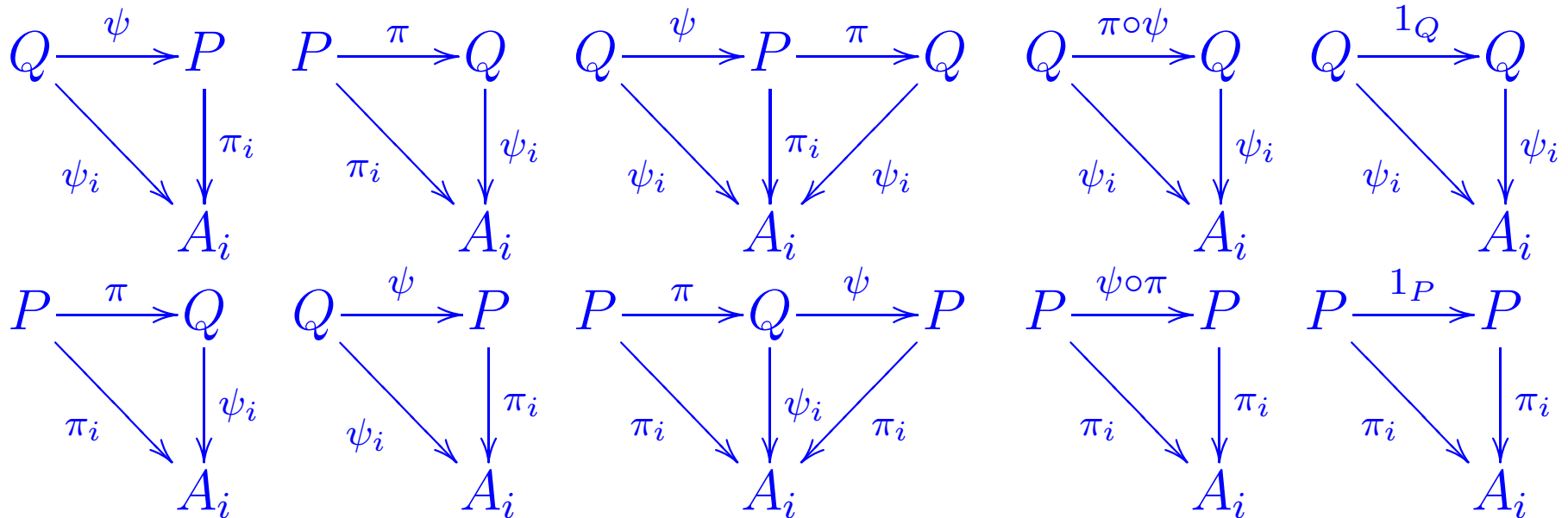


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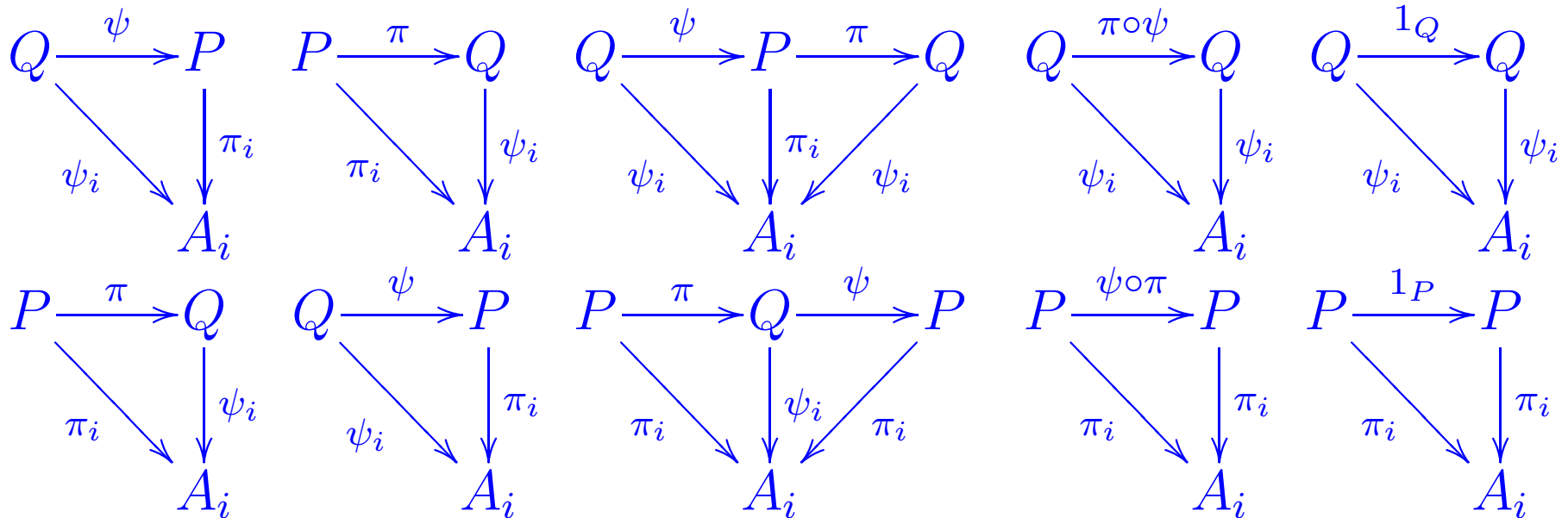


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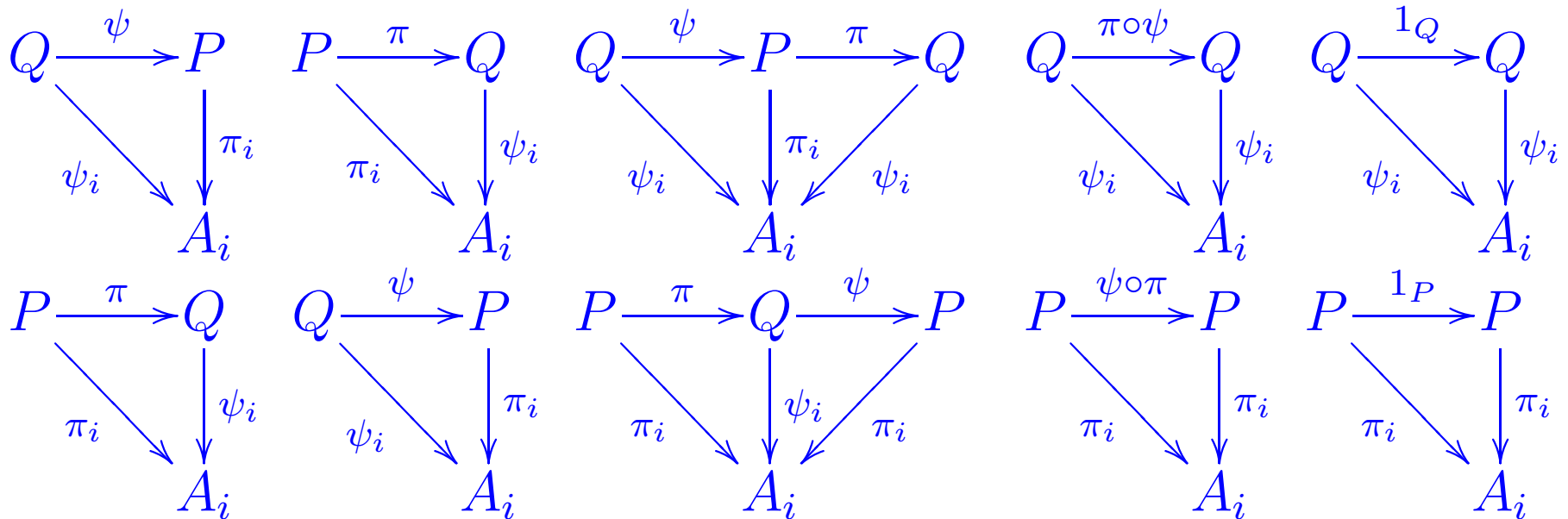


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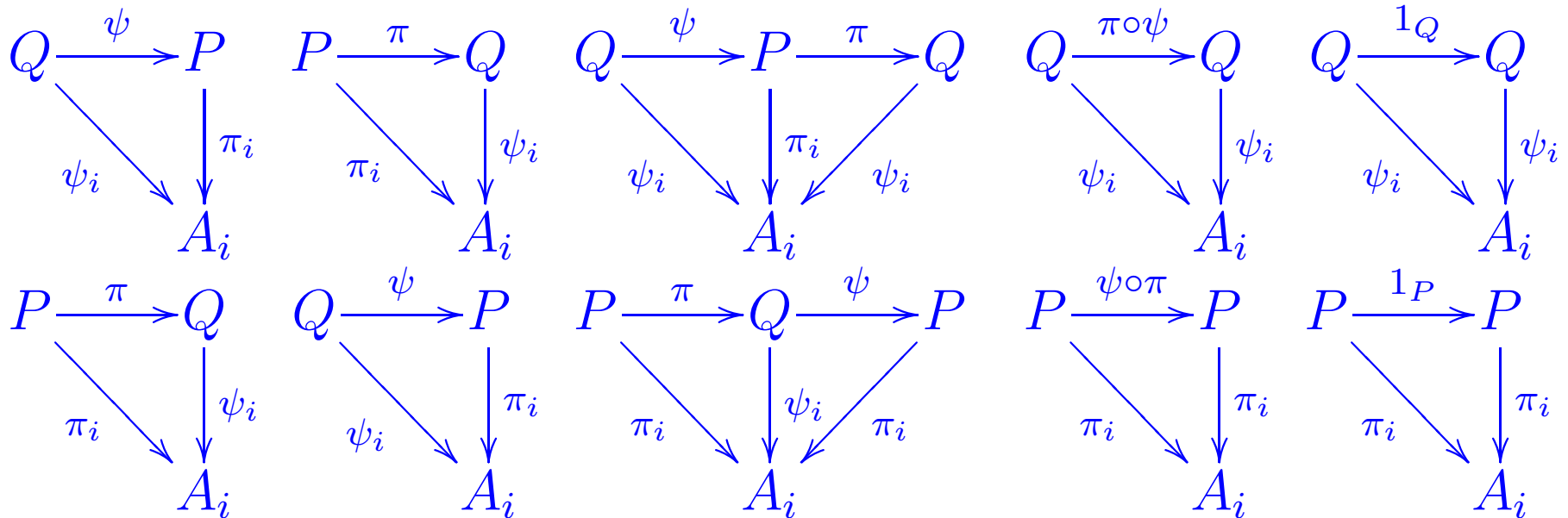


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*This is a very good practice. Please follow the proof of Theorem (7.3) and give a proof of Theorem (7.5).*

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**Proof.** We already have these two commutative diagrams

$$\begin{array}{ccc} F & \xrightarrow{\bar{g}_1} & F' \\ \uparrow i & & \uparrow i' \\ X & \xrightarrow{f} & X' \end{array} \quad \begin{array}{ccc} F' & \xrightarrow{\bar{g}_2} & F \\ \uparrow i' & & \uparrow i \\ X' & \xrightarrow{f^{-1}} & X \end{array}$$

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**Proof.** Note that  $i \circ f^{-1} \circ f = i$ ,

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Let  $F$  and  $F'$  be objects in a concrete category  $\mathcal{C}$ . Suppose that  $F$  is free on the set  $X$  and  $F'$  is free on the set  $X'$ . If  $|X| = |X'|$ , then  $F$  and  $F'$  are equivalent.

**Proof.** Note that  $i \circ f^{-1} \circ f = i$ , so we have

$$\begin{array}{cccc}
 F \xrightarrow{\bar{g}_1} F' & F' \xrightarrow{\bar{g}_2} F & F \xrightarrow{\bar{g}_1} F' \xrightarrow{\bar{g}_2} F & F \xrightarrow{\bar{g}_2 \circ \bar{g}_1} F \\
 \uparrow i & \uparrow i' & \uparrow i & \uparrow i \\
 X \xrightarrow{f} X' & X' \xrightarrow{f^{-1}} X & X \xrightarrow{f} X' \xrightarrow{f^{-1}} X & X \xrightarrow{i \circ f^{-1} \circ f} X
 \end{array}$$

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 F & \xrightarrow{\bar{g}_1} & F' & & F' & \xrightarrow{\bar{g}_2} & F & & F & \xrightarrow{\bar{g}_1} & F' & \xrightarrow{\bar{g}_2} & F & & F & \xrightarrow{\bar{g}_2 \circ \bar{g}_1} & F \\
 \uparrow i & & \uparrow i' & & \uparrow i' & & \uparrow i & & \uparrow i & & \uparrow i' & & \uparrow i & & \uparrow i & & \nearrow i \\
 X & \xrightarrow{f} & X' & & X' & \xrightarrow{f^{-1}} & X & & X & \xrightarrow{f} & X' & \xrightarrow{f^{-1}} & X & & X & & 
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$$\begin{array}{ccccc}
 F & \xrightarrow{\bar{g}_1} & F' & & F' & \xrightarrow{\bar{g}_2} & F & & F & \xrightarrow{\bar{g}_1} & F' & \xrightarrow{\bar{g}_2} & F & & F & \xrightarrow{\bar{g}_2 \circ \bar{g}_1} & F \\
 \uparrow i & & \uparrow i' & & \uparrow i' & & \uparrow i & & \uparrow i & & \uparrow i' & & \uparrow i & & \uparrow i & \nearrow i & \\
 X & \xrightarrow{f} & X' & & X' & \xrightarrow{f^{-1}} & X & & X & \xrightarrow{f} & X' & \xrightarrow{f^{-1}} & X & & X & & 
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 \uparrow i & & \uparrow i' & & \uparrow i' & & \uparrow i & & \uparrow i & & \uparrow i' & & \uparrow i & & \uparrow i & & \nearrow i & & \uparrow i & & \nearrow i \\
 X & \xrightarrow{f} & X' & & X' & \xrightarrow{f^{-1}} & X & & X & \xrightarrow{f} & X' & \xrightarrow{f^{-1}} & X & & X & & & & X & & & & 
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 \uparrow i & & \uparrow i' & & \uparrow i' & & \uparrow i & & \uparrow i & & \uparrow i' & & \uparrow i & & \uparrow i & & \nearrow i & & \uparrow i & & \nearrow i \\
 X & \xrightarrow{f} & X' & & X' & \xrightarrow{f^{-1}} & X & & X & \xrightarrow{f} & X' & \xrightarrow{f^{-1}} & X & & X & & & & X & & & & 
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 F \xrightarrow{\overline{g_1}} F' & F' \xrightarrow{\overline{g_2}} F & F \xrightarrow{\overline{g_1}} F' \xrightarrow{\overline{g_2}} F & F \xrightarrow{\overline{g_2 \circ g_1}} F & F \xrightarrow{1_F} F \\
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By the uniqueness of the morphism  $F \rightarrow F$  which makes the diagram work, we have  $\overline{g_2} \circ \overline{g_1} = 1_F$ .

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 F \xrightarrow{\overline{g_1}} F' & F' \xrightarrow{\overline{g_2}} F & F \xrightarrow{\overline{g_1}} F' \xrightarrow{\overline{g_2}} F & F \xrightarrow{\overline{g_2 \circ g_1}} F & F \xrightarrow{1_F} F \\
 \uparrow i & \uparrow i' & \uparrow i & \uparrow i & \uparrow i \\
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 \uparrow i & \uparrow i' & \uparrow i & \uparrow i & \uparrow i \\
 X \xrightarrow{f} X' & X' \xrightarrow{f^{-1}} X & X \xrightarrow{f} X' \xrightarrow{f^{-1}} X & X & X
 \end{array}$$

By the uniqueness of the morphism  $F \rightarrow F$  which makes the diagram work, we have  $\overline{g_2} \circ \overline{g_1} = 1_F$ . With a similar discussion, we can also show that  $\overline{g_1} \circ \overline{g_2} = 1_{F'}$ .

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 \uparrow i & \uparrow i' & \uparrow i & \uparrow i & \uparrow i \\
 X \xrightarrow{f} X' & X' \xrightarrow{f^{-1}} X & X \xrightarrow{f} X' \xrightarrow{f^{-1}} X & X & X \\
 & & & \nearrow i & \nearrow i
 \end{array}$$

By the uniqueness of the morphism  $F \rightarrow F$  which makes the diagram work, we have  $\bar{g}_2 \circ \bar{g}_1 = 1_F$ . With a similar discussion, we can also show that  $\bar{g}_1 \circ \bar{g}_2 = 1_{F'}$ . Hence,  $F$  and  $F'$  are equivalent.