Modern Algebra I

Lecture 9

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2009, Fall

Today, we will finish the section

Section II.1: Free Abelian Groups

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and start to cover the section

Section II.2: Finitely Generated Abelian Groups

Chapter II: THE STRUCTURE OF GROUPS

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Definition. An abelian group F is called a **free** abelian group if F has a basis.

Review

Example. $\sum_{i\in I}\mathbb{Z}$ is a free abelian group with a basis $\{\mathbf{e}_i \mid i \in I\}$ where for each $k \in I$, $\mathbf{e}_k = (a_i)_{i\in I}$ denotes the element in $\sum_{i\in I}\mathbb{Z}$ such that

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Theorem. Let F be a free abelian group with a nonempty basis X. Then every nonzero element of F can be written uniquely in the form $n_1 x_1 + \cdots + n_k x_k$ with $n_i \in \mathbb{Z} \setminus \{0\}$ and $x_1, \ldots, x_k \in X$ distinct.

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Remark. If F is a free abelian group with a nonempty basis X, then F is free on X in the category of abelian groups.

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Moreover, if
$k_1x_1 + \cdots + k_{j-1}x_{j-1} + k_j(x_j + ax_i) + k_{j+1}x_{j+1} + \ldots + k_nx_n = 0$,
with $k_1, \ldots, k_n \in \mathbb{Z}$, then
$k_1x_1 + \cdots + (k_i + ak_j)x_i + \cdots + k_jx_j + \ldots + k_nx_n = 0$; since $\{x_1, \ldots, x_n\}$ is linearly independent, we have
$k_1 = \cdots = k_i + ak_j = \cdots = k_j = \cdots = k_n = 0$ and this implies
$k_1 = \cdots = k_i = \cdots = k_j = \cdots = k_n = 0$. Hence Y is linearly independent and so (2) is proved.

An Easy-but-useful Lemma

Lemma (1.5). If $\{x_1, \ldots, x_n\}$ is a basis of a free abelian group F and if $a \in \mathbb{Z}$, then for all $i \neq j$, $\{x_1, \ldots, x_{j-1}, x_j + ax_i, x_{j+1}, \ldots, x_n\}$ is also a basis of F.

Proof. Let $Y = \{x_1, \ldots, x_{j-1}, x_j + ax_i, x_{j+1}, \ldots, x_n\}$.

We need to show that (1) Y generates F *(Proved!)*

and (2) Y is linearly independent. *(Proved!)*

Moreover, if

$k_1x_1+\cdots+k_{j-1}x_{j-1}+k_j(x_j+ax_i)+k_{j+1}x_{j+1}+\ldots+k_nx_n = 0$,

with $k_1, \ldots, k_n \in \mathbb{Z}$, then

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Theorem (1.6)

If F is a free abelian group of finite rank n and if G is a nonzero subgroup of F, then there exists a basis $\{x_1, \ldots, x_n\}$ of F, an integer r with $1 \leq r \leq n$ and positive integers $d_1, \ldots, d_r$ such that $d_1 \mid d_2 \mid \cdots \mid d_r$ and $\{d_1 x_1, \ldots, d_r x_r\}$ is a basis of G.

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Proof. We will prove this theorem by induction on n.

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Proof. We will prove this theorem by induction on n. For the base case $n = 1$, $F = \langle x_1 \rangle$ for some $x_1 \in F$.

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Proof of Theorem (1.6)

Theorem (1.6)**.** If F is a free abelian group of finite rank n and if G is a nonzero subgroup of F, then there exists a basis $\{x_1, \ldots, x_n\}$ of F, an integer r with $1 \leq r \leq n$ and positive integers $d_1, \ldots, d_r$ such that $d_1 \mid d_2 \mid \cdots \mid d_r$ and $\{d_1 x_1, \ldots, d_r x_r\}$ is a basis of G.

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Since H is a free abelian group of rank $n-1$ and since $G \cap H$ is a nonzero subgroup of H, by the induction hypothesis, there exists a basis $\{x_2, \ldots, x_n\}$ of H, an integer r with $2 \leq r \leq n$ and positive integers $d_2, \ldots, d_r$ such that $d_2 \mid d_3 \mid \cdots \mid d_r$ and $\{d_2x_2, \ldots, d_rx_r\}$ is a basis of $G \cap H$. Since $F = \langle x_1 \rangle \oplus H$,

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Since H is a free abelian group of rank $n-1$ and since $G \cap H$ is a nonzero subgroup of H, by the induction hypothesis, there exists a basis $\{x_2, \ldots, x_n\}$ of H, an integer r with $2 \leq r \leq n$ and positive integers $d_2, \ldots, d_r$ such that $d_2 \mid d_3 \mid \cdots \mid d_r$ and $\{d_2x_2, \ldots, d_rx_r\}$ is a basis of $G \cap H$. Since $F = \langle x_1 \rangle \oplus H$, $\{x_1, x_2, \ldots, x_n\}$ is a basis of F;

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Since H is a free abelian group of rank $n - 1$ and since $G \cap H$ is a nonzero subgroup of H, by the induction hypothesis, there exists a basis $\{x_2, \ldots, x_n\}$ of H, an integer r with $2 \leq r \leq n$ and positive integers $d_2, \ldots, d_r$ such that $d_2 \mid d_3 \mid \cdots \mid d_r$ and $\{d_2 x_2, \ldots, d_r x_r\}$ is a basis of $G \cap H$. Since $F = \langle x_1 \rangle \oplus H$, $\{x_1, x_2, \ldots, x_n\}$ is a basis of F; also, since $G = \langle d_1 x_1 \rangle \oplus (G \cap H)$,

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Since H is a free abelian group of rank $n-1$ and since $G \cap H$ is a nonzero subgroup of H, by the induction hypothesis, there exists a basis $\{x_2, \dots, x_n\}$ of H, an integer r with $2 \leq r \leq n$ and positive integers $d_2, \dots, d_r$ such that $d_2 \mid d_3 \mid \cdots \mid d_r$ and $\{d_2x_2, \dots, d_rx_r\}$ is a basis of $G \cap H$. Since $F = \langle x_1 \rangle \oplus H$, $\{x_1, x_2, \dots, x_n\}$ is a basis of F; also, since $G = \langle d_1x_1 \rangle \oplus (G \cap H)$, $\{d_1x_1, d_2x_2, \dots, d_rx_r\}$ is a basis of G. Hence it remains to show that $d_1 \mid d_2$.

Proof of Theorem (1.6)

Theorem (1.6)**.** If F is a free abelian group of finite rank n and if G is a nonzero subgroup of F, then there exists a basis $\{x_1, \ldots, x_n\}$ of F, an integer r with $1 \leq r \leq n$ and positive integers $d_1, \ldots, d_r$ such that $d_1 \mid d_2 \mid \cdots \mid d_r$ and $\{d_1x_1, \ldots, d_rx_r\}$ is a basis of G.

Since H is a free abelian group of rank $n-1$ and since $G \cap H$ is a nonzero subgroup of H, by the induction hypothesis, there exists a basis $\{x_2, \ldots, x_n\}$ of H, an integer r with $2 \leq r \leq n$ and positive integers $d_2, \ldots, d_r$ such that $d_2 \mid d_3 \mid \cdots \mid d_r$ and $\{d_2x_2, \ldots, d_rx_r\}$ is a basis of $G \cap H$. Since $F = \langle x_1 \rangle \oplus H$, $\{x_1, x_2, \ldots, x_n\}$ is a basis of F; also, since $G = \langle d_1x_1 \rangle \oplus (G \cap H)$, $\{d_1x_1, d_2x_2, \ldots, d_rx_r\}$ is a basis of G. Hence it remains to show that $d_1 \mid d_2$.

Proof of Theorem (1.6)

- S contains all possible coefficients for elements in G.
- d_1 is the least positive integers in S.
- $\{x_1, x_2, \ldots, x_n\}$ is a basis of F and $\{d_1x_1, d_2x_2, \ldots, d_rx_r\}$ is a basis of G with $d_2 \mid d_3 \mid \cdots \mid d_n$.
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Exercise for Section II.1

1, 2, 6, 7, 9, 10.

Chapter II

The Structure of Groups

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Section II.2: Finitely Generated Abelian Groups

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Theorem (2.1)**.** Every finitely generated abelian group G

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Theorem (2.1)**.** Every finitely generated abelian group G is (isomorphic to) a finite direct sum of cyclic groups in which the finite cyclic summands (if any) are of orders $m_1, \ldots, m_t$, where $m_1 > 1$ and $m_1 \mid m_2 \mid \cdots \mid m_t$.

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Proof. If $G = 0$, nothing needs to be proved.

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Proof. Assume $G \neq 0$ and G is generated by n elements. By Theorem (1.4) there is a group epimorphism $\pi : F \to G$ with F a free abelian group of rank n. Consider $\operatorname{Ker} \pi$.

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Theorem (2.1)**.** Every finitely generated abelian group G is (isomorphic to) a finite direct sum of cyclic groups in which the finite cyclic summands (if any) are of orders $m_1, \ldots, m_t$, where $m_1 > 1$ and $m_1 \mid m_2 \mid \cdots \mid m_t$.

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Theorem (1.6). If F is a free abelian group of finite rank n and if G is a nonzero subgroup of F, then there exists a basis $\{x_1, \ldots, x_n\}$ of F,

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$$G \cong F/\operatorname{Ker}\pi$$

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$$G \;\cong\; F/\operatorname{Ker}\pi \cong \textstyle\sum_{i=1}^{n} \langle x_i \rangle \big/ \sum_{i=1}^{n} \langle d_i x_i \rangle$$

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$$\begin{aligned} G &\cong F / \operatorname{Ker} \pi \cong \textstyle\sum_{i=1}^n \langle x_i \rangle \big/ \sum_{i=1}^n \langle d_i x_i \rangle \\ &\cong \textstyle\sum_{i=1}^n \langle x_i \rangle / \langle d_i x_i \rangle \end{aligned}$$

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- If $d_i = 1$, then $\mathbb{Z}/d_i\mathbb{Z} = \mathbb{Z}/\mathbb{Z} = 0$;

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Proof.

$$G \cong F/\operatorname{Ker}\pi \cong \sum_{i=1}^{n}\langle x_i\rangle \Big/ \sum_{i=1}^{n}\langle d_i x_i\rangle$$
$$\cong \sum_{i=1}^{n}\langle x_i\rangle/\langle d_i x_i\rangle \cong \sum_{i=1}^{n}\mathbb{Z}/d_i\mathbb{Z}.$$

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- if $d_i = 0$,

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Proof.

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- If $d_i = 1$, then $\mathbb{Z}/d_i\mathbb{Z} = \mathbb{Z}/\mathbb{Z} = 0$;
- if $d_i = 0$, then $\mathbb{Z}/d_i\mathbb{Z} = \mathbb{Z}/0 \cong \mathbb{Z}$;

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Proof.

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Proof.

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- if $d_i > 1$, then $\mathbb{Z}/d_i\mathbb{Z} \cong \mathbb{Z}_{d_i}$.

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Proof.

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Let $m_1, \ldots, m_t$ be those d_i, in order, such that $d_i > 1$ and let s be the number of d_i such that $d_i = 0$.

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Proof.
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- If $d_i = 1$, then $\mathbb{Z}/d_i\mathbb{Z} = \mathbb{Z}/\mathbb{Z} = 0$;
- if $d_i = 0$, then $\mathbb{Z}/d_i\mathbb{Z} = \mathbb{Z}/0 \cong \mathbb{Z}$;
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Let $m_1, \ldots, m_t$ be those d_i, in order, such that $d_i > 1$ and let s be the number of d_i such that $d_i = 0$. Then

$$G \cong \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_t} \oplus \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{s \text{ summands}},$$

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$$\begin{aligned} G &\cong F/\operatorname{Ker}\pi \cong \sum_{i=1}^{n} \langle x_i \rangle \Big/ \sum_{i=1}^{n} \langle d_i x_i \rangle \\ &\cong \sum_{i=1}^{n} \langle x_i \rangle / \langle d_i x_i \rangle \cong \sum_{i=1}^{n} \mathbb{Z}/d_i\mathbb{Z}. \end{aligned}$$

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Let $m_1, \ldots, m_t$ be those d_i, in order, such that $d_i > 1$ and let s be the number of d_i such that $d_i = 0$. Then
$$G \cong \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_t} \oplus \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{s \text{ summands}},$$
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Proof.
$$G \cong F/\operatorname{Ker}\pi \cong \sum_{i=1}^{n}\langle x_i\rangle \Big/ \sum_{i=1}^{n}\langle d_i x_i\rangle \cong \sum_{i=1}^{n}\langle x_i\rangle/\langle d_i x_i\rangle \cong \sum_{i=1}^{n}\mathbb{Z}/d_i\mathbb{Z}.$$

- If $d_i = 1$, then $\mathbb{Z}/d_i\mathbb{Z} = \mathbb{Z}/\mathbb{Z} = 0$;
- if $d_i = 0$, then $\mathbb{Z}/d_i\mathbb{Z} = \mathbb{Z}/0 \cong \mathbb{Z}$;
- if $d_i > 1$, then $\mathbb{Z}/d_i\mathbb{Z} \cong \mathbb{Z}_{d_i}$.

Let $m_1, \ldots, m_t$ be those d_i, in order, such that $d_i > 1$ and let s be the number of d_i such that $d_i = 0$. Then
$$G \cong \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_t} \oplus \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{s \text{ summands}},$$
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Remark

Theorem (2.1)**.** Every finitely generated abelian group G is (isomorphic to) a finite direct sum of cyclic groups in which the finite cyclic summands (if any) are of orders $m_1, \ldots, m_t$, where $m_1 > 1$ and $m_1 \mid m_2 \mid \cdots \mid m_t$.

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The key of proving Theorem (2.2) is Theorem (2.1) and the following Lemma.

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Proof. For (i), let $mu, mv \in mG$ with $u, v \in G$.

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Next, we see and prove some properties related to these subgroups.

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There are two more statements in Lemma (2.5). Before we state and prove them, we see two easy lemmas.