Modern Algebra I

Lecture 8

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Section II.1: Free Abelian Groups

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THE STRUCTURE OF GROUPS

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Definition. An abelian group F is called a **free** abelian group if F has a basis.

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Remark. In Theorem (1.1), we show that all free abelian groups are of the form $\sum_{i \in I} \mathbb{Z}$, up to isomorphism.

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Proof. We have shown (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii)$\Rightarrow$ (i) $\Rightarrow$ (iv) $\Rightarrow$ (iii).

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(i) F has a nonempty basis.

(iv) F is a free object in the category of abelian groups, i.e., there exists a nonempty set X and a function $\iota : X \to F$ with the following property: given an abelian group G and function $f : X \to G$, there exists a unique homomorphism of groups $\bar{f} : F \to G$ such that $\bar{f}\iota = f$.

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Proof of (i) $\Rightarrow$ (iv)

Proof. Let X be a nonempty basis of F and let $\iota : X \to F$ be the inclusion map. Let $f : X \to G$ be a function. We want to show that there exists a unique group homomorphism $\bar{f} : F \to G$ such that $\bar{f}\iota = f$. Since every nonzero element in F can be written uniquely in the form $n_1x_1 + \cdots + n_kx_k$ with $n_i \in \mathbb{Z} \setminus \{0\}$ and $x_1, \ldots, x_k \in X$ distinct, we can define a map $\bar{f} : F \to G$ as $\bar{f}(0) = 0$ and $\bar{f}(n_1x_1 + \cdots + n_kx_k) = n_1f(x_1) + \cdots + n_kf(x_k)$. Since G is abelian, it is easy to see that $\bar{f}$ is a homomorphism. It is also easy to see that $\bar{f}\iota = f$, i.e., $\bar{f}(x) = f(x)\ \forall x \in X$. Suppose that $g : F \to G$ is a group homomorphism such that $g\iota = f$.

Proof of (i) $\Rightarrow$ (iv)

Proof. Let X be a nonempty basis of F and let $\iota : X \to F$ be the inclusion map. Let $f : X \to G$ be a function. We want to show that there exists a unique group homomorphism $\bar{f} : F \to G$ such that $\bar{f}\iota = f$. Since every nonzero element in F can be written uniquely in the form $n_1x_1 + \cdots + n_kx_k$ with $n_i \in \mathbb{Z} \setminus \{0\}$ and $x_1, \ldots, x_k \in X$ distinct, we can define a map $\bar{f} : F \to G$ as $\bar{f}(0) = 0$ and $\bar{f}(n_1x_1 + \cdots + n_kx_k) = n_1f(x_1) + \cdots + n_kf(x_k)$. Since G is abelian, it is easy to see that $\bar{f}$ is a homomorphism. It is also easy to see that $\bar{f}\iota = f$, i.e., $\bar{f}(x) = f(x)\ \forall x \in X$. Suppose that $g : F \to G$ is a group homomorphism such that $g\iota = f$. For every nonzero element $n_1x_1 + \cdots + n_kx_k \in F$,

Proof of (i) $\Rightarrow$ (iv)

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Proof of (i) $\Rightarrow$ (iv)

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$$g(n_1x_1 + \cdots + n_kx_k) = n_1g(x_1) + \cdots + n_kg(x_k)$$
$$= n_1f(x_1) + \cdots + n_kf(x_k)$$

Proof of (i) $\Rightarrow$ (iv)

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Proof of (i) $\Rightarrow$ (iv)

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$g(n_1x_1 + \cdots + n_kx_k) = n_1g(x_1) + \cdots + n_kg(x_k)$
$= n_1f(x_1) + \cdots + n_kf(x_k) = \bar{f}(n_1x_1 + \cdots + n_kx_k)$. Hence, $g = \bar{f}$

Proof of (i) $\Rightarrow$ (iv)

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$g(n_1x_1 + \cdots + n_kx_k) = n_1g(x_1) + \cdots + n_kg(x_k)$
$= n_1f(x_1) + \cdots + n_kf(x_k) = \bar{f}(n_1x_1 + \cdots + n_kx_k)$. Hence, $g = \bar{f}$ and this shows the uniqueness.

Proof of (i) $\Rightarrow$ (iv)

Proof. Let X be a nonempty basis of F and let $\iota : X \to F$ be the inclusion map. Let $f : X \to G$ be a function. We want to show that there exists a unique group homomorphism $\bar{f} : F \to G$ such that $\bar{f}\iota = f$.

Remark. We just prove that if F is a free abelian group with a nonempty basis X,

Proof of (i) $\Rightarrow$ (iv)

Proof. Let X be a nonempty basis of F and let $\iota : X \longrightarrow F$ be the inclusion map. Let $f : X \longrightarrow G$ be a function. We want to show that there exists a unique group homomorphism $\bar{f} : F \longrightarrow G$ such that $\bar{f}\iota = f$.

Remark. We just prove that if F is a free abelian group with a nonempty basis X, then F is free on X in the category of abelian groups.

Theorem (1.1)

The following conditions on an abelian group F are equivalent:

(i) F has a nonempty basis.

(ii) F is the (internal) direct sum of a family of infinite cyclic subgroups.

(iii) F is (isomorphic to) a direct sum of copies of the additive groups $\mathbb{Z}$ of integers, i.e., $F \cong \sum_{i \in I} \mathbb{Z}$ for some index set I.

(iv) F is a free object in the category of abelian groups, i.e., there exists a nonempty set X and a function $\iota : X \to F$ with the following property: given an abelian group G and function $f : X \to G$, there exists a unique homomorphism of groups $\bar{f} : F \to G$ such that $\bar{f}\iota = f$.

Proof. We have shown (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii)$\Rightarrow$ (i) $\Rightarrow$ (iv) $\Rightarrow$ (iii).

Proof of (iv) $\Rightarrow$ (iii)

(iii) F is (isomorphic to) a direct sum of copies of the additive groups $\mathbb{Z}$ of integers, i.e., $F \cong \sum_{i \in I} \mathbb{Z}$ for some index set I.

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Proof.

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Proof. Construct the free abelian group $\sum_{x \in X} \mathbb{Z}$.

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Proof. Construct the free abelian group $\sum_{x\in X} \mathbb{Z}$. Since $Y = \{\mathbf{e}_x \mid x \in X\}$ is a basis of $\sum_{x\in X} \mathbb{Z}$,

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Proof. Construct the free abelian group $\sum_{x \in X} \mathbb{Z}$. Since $Y = \{\mathbf{e}_x \mid x \in X\}$ is a basis of $\sum_{x \in X} \mathbb{Z}$, $\sum_{x \in X} \mathbb{Z}$ is free on the set Y in the category of abelian groups.

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Proof. Construct the free abelian group $\sum_{x \in X} \mathbb{Z}$. Since $Y = \{\mathbf{e}_x \mid x \in X\}$ is a basis of $\sum_{x \in X} \mathbb{Z}$, $\sum_{x \in X} \mathbb{Z}$ is free on the set Y in the category of abelian groups. Note that $|X| = |Y|$. By Theorem (I.7.8), (If F and F' are free objects in a concrete category $\mathcal{C}$ such that F is free on the set X and F' is free on the set X',

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Proof. Construct the free abelian group $\sum_{x \in X} \mathbb{Z}$. Since $Y = \{\mathbf{e}_x \mid x \in X\}$ is a basis of $\sum_{x \in X} \mathbb{Z}$, $\sum_{x \in X} \mathbb{Z}$ is free on the set Y in the category of abelian groups. Note that $|X| = |Y|$. By Theorem (I.7.8), (If F and F' are free objects in a concrete category $\mathcal{C}$ such that F is free on the set X and F' is free on the set X', then $|X| = |X'|$ implies that F and F' are equivalent.)

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Proof. Construct the free abelian group $\sum_{x \in X} \mathbb{Z}$. Since $Y = \{\mathbf{e}_x \mid x \in X\}$ is a basis of $\sum_{x \in X} \mathbb{Z}$, $\sum_{x \in X} \mathbb{Z}$ is free on the set Y in the category of abelian groups. Note that $|X| = |Y|$. By Theorem (I.7.8), F and $\sum_{x \in X} \mathbb{Z}$ are equivalent in the category of abelian groups,

Proof of (iv) $\Rightarrow$ (iii)

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Proof. Construct the free abelian group $\sum_{x \in X} \mathbb{Z}$. Since $Y = \{\mathbf{e}_x \mid x \in X\}$ is a basis of $\sum_{x \in X} \mathbb{Z}$, $\sum_{x \in X} \mathbb{Z}$ is free on the set Y in the category of abelian groups. Note that $|X| = |Y|$. By Theorem (I.7.8), F and $\sum_{x \in X} \mathbb{Z}$ are equivalent in the category of abelian groups, i.e., they are isomorphic as groups.

Theorem (1.1)

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(iv) F is a free object in the category of abelian groups, i.e., there exists a nonempty set X and a function $\iota : X \to F$ with the following property: given an abelian group G and function $f : X \to G$, there exists a unique homomorphism of groups $\bar{f} : F \to G$ such that $\bar{f}\iota = f$.

Proof. We have shown (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii)$\Rightarrow$ (i) $\Rightarrow$ (iv) $\Rightarrow$ (iii).

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- Case 2: Every basis of F is infinite.

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Therefore, $|F/2F| = 2^n$.

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This completes the proof.

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- If X is infinite, by Lemma (1.2.A), every basis X' of F is infinite. By Lemma (1.2.B), $|X| = |F| = |X'|$.

Hence in either case, any two bases of F have the same cardinality.

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Let F_1 and F_2 be free abelian groups.

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$$F_1 \cong F_2 \iff F_1 \text{ and } F_2 \text{ have the same rank.}$$

Proof. "$\Leftarrow$": Let X_1 be a basis of F_1 and let X_2 be a basis of F_2. Then F_1 is a free object on X_1 and F_2 is a free object on X_2 in the category $\mathcal{A}$ of abelian groups. Moreover, since F_1 and F_2 have the same rank, $|X_1| = |X_2|$. By Theorem (I.7.8), F_1 and F_2 are equivalent in $\mathcal{A}$, i.e., F_1 and F_2 are isomorphic as groups.

"$\Rightarrow$": Let $\varphi : F_1 \to F_2$ be a group isomorphism and let X be a basis of F_1. If we show that $\varphi(X)$ is a basis of F_2, then we have $|X| = |\varphi(X)|$ and this implies F_1 and F_2 have the same rank.

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Therefore, $G = \operatorname{Im} \bar{f}$.

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Remark. Theorem (1.6) shows that if F is a free abelian group of finite rank n,

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Remark. Theorem (1.6) shows that if F is a free abelian group of finite rank n, then every subgroup of F is free abelian of finite rank $\leq n$.

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Remark. Theorem (1.6) shows that if F is a free abelian group of finite rank n, then every subgroup of F is free abelian of finite rank $\leq n$. In fact, if F is a free abelian group of rank α,

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Remark. Theorem (1.6) shows that if F is a free abelian group of finite rank n, then every subgroup of F is free abelian of finite rank $\leq n$. In fact, if F is a free abelian group of rank α, not necessarily finite,

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Remark. Theorem (1.6) shows that if F is a free abelian group of finite rank n, then every subgroup of F is free abelian of finite rank $\leq n$.

We will use this Remark to prove the following Corollary.

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If F is a free abelian group of finite rank n and if G is a nonzero subgroup of F, then there exists a basis $\{x_1, \ldots, x_n\}$ of F, an integer r with $1 \leq r \leq n$ and positive integers $d_1, \ldots, d_r$ such that $d_1 \mid d_2 \mid \cdots \mid d_r$ and $\{d_1 x_1, \ldots, d_r x_r\}$ is a basis of G.

Remark. Theorem (1.6) shows that if F is a free abelian group of finite rank n, then every subgroup of F is free abelian of finite rank $\leq n$.

We will use this Remark to prove the following Corollary.

Corollary (1.7)**.** If G is a finitely generated abelian group generated by n elements,

Theorem (1.6)

If F is a free abelian group of finite rank n and if G is a nonzero subgroup of F, then there exists a basis $\{x_1, \ldots, x_n\}$ of F, an integer r with $1 \leq r \leq n$ and positive integers $d_1, \ldots, d_r$ such that $d_1 \mid d_2 \mid \cdots \mid d_r$ and $\{d_1x_1, \ldots, d_rx_r\}$ is a basis of G.

Remark. Theorem (1.6) shows that if F is a free abelian group of finite rank n, then every subgroup of F is free abelian of finite rank $\leq n$.

We will use this Remark to prove the following Corollary.

Corollary (1.7). If G is a finitely generated abelian group generated by n elements, then every subgroup H of G can be generated by m elements with $m \leq n$.

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Proof. Since G may be generated by n elements, by Theorem (1.4), there exists a group epimorphism $\varphi : F \to G$ where F is a free abelian group of rank n.

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