Modern Algebra I

Lecture 7

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Today, we will continue the section

Section I.8: Direct Products and Direct Sums

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and start to cover the section

Section II.1: Free Abelian Groups

Chapter I

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Recall that $K \le G$ and $N \lhd G \implies \langle K \cup N \rangle = KN \le G$.

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Theorem. Let H and K be subgroups of a group G. Suppose

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- $H \lhd G$ and $K \lhd G$,
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In Theorem (5.3) we show that

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$\Rightarrow hk = kh, \forall h \in H$ and $\forall k \in K$.

Proof. Consider the map $f : H \times K \to G$ defined by $(h,k) \mapsto hk$. Then f is a group homomorphism because

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Therefore, f is one-to-one and thus f is an isomorphism.

Proof of Theorem (8.6)

Theorem (8.6)**.** Let $\{N_i \mid i \in I\}$ be a family of normal subgroups of a group G. Suppose

- $G = \langle \bigcup_{i \in I} N_i \rangle$,
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Proof. *I need to remind you that you won't have time to take notes.*

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Proof. All elements in N_j commute with all element in N_k. For every $(a_i)_{i \in I} \in \prod_{i \in I}^{W} N_i$, the set $I_{\neq e} = \{i \in I \mid a_i \neq e\}$ is finite.

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Proof. All elements in N_j commute with all element in N_k. For every $(a_i)_{i \in I} \in \prod_{i \in I}^{W} N_i$, the set $I_{\neq e} = \{i \in I \mid a_i \neq e\}$ is finite. Moreover, by our above observation, the product of the elements a_i with $i \in I_{\neq e}$ does not dependent on the order of the elements a_i,

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Hence the map $f : \prod_{i \in I}^{W} N_i \to G$, defined by $(e)_{i \in I} \mapsto e$ and $(a_i)_{i \in I} \mapsto \prod_{i \in I_{\neq e}} a_i$ for all non-identity $(a_i)_{i \in I} \in \prod_{i \in I}^{W} N_i$,

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Proof of Theorem (8.6)

All elements in N_j commute with all element in N_k, $\forall j \neq k$.

The map $f : \prod_{i\in I}^{W} N_i \to G$, defined by $(e)_{i\in I} \mapsto e$ and $(a_i)_{i\in I} \mapsto \prod_{i\in I_{\neq e}} a_i$, is a group homomorphism.

Suppose that f is not one-to-one.

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Suppose that f is not one-to-one.

Then there exists $(a_i)_{i \in I} \in \operatorname{Ker} f$ with

$I_{\neq e} = \{i_1, i_2, \ldots, i_\ell\} \neq \varnothing$.

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All elements in N_j commute with all element in N_k, $\forall j \neq k$.

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Suppose that f is not one-to-one.

Then there exists $(a_i)_{i\in I} \in \operatorname{Ker} f$ with $I_{\neq e} = \{i_1, i_2, \ldots, i_\ell\} \neq \varnothing$. Hence $a_{i_1} a_{i_2} \cdots a_{i_\ell} = e$ and this implies $a_{i_2} \cdots a_{i_\ell} = a_{i_1}^{-1} \in N_{i_1} \cap \langle \bigcup_{i \neq i_1} N_i \rangle = \{e\}$; hence $a_{i_1} = e$ and this contradicts the definition of $I_{\neq e}$.

Proof of Theorem (8.6)

All elements in N_j commute with all element in N_k, $\forall j \neq k$.
The map $f : \prod_{i\in I}^{W} N_i \to G$, defined by $(e)_{i\in I} \mapsto e$ and $(a_i)_{i\in I} \mapsto \prod_{i\in I_{\neq e}} a_i$, is a group homomorphism.

Suppose that f is not one-to-one.
Then there exists $(a_i)_{i\in I} \in \operatorname{Ker} f$ with $I_{\neq e} = \{i_1, i_2, \ldots, i_\ell\} \neq \varnothing$. Hence $a_{i_1} a_{i_2} \cdots a_{i_\ell} = e$ and this implies $a_{i_2} \cdots a_{i_\ell} = a_{i_1}^{-1} \in N_{i_1} \cap \langle \bigcup_{i\neq i_1} N_i \rangle = \{e\}$; hence $a_{i_1} = e$ and this contradicts the definition of $I_{\neq e}$. Therefore, f is a monomorphism.

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Then $f\big((a_i)_{i\in I}\big) = a_{i_1} a_{i_2} \cdots a_{i_\ell} = a$. Hence f is onto and thus f is an isomorphism.

Internal Weak Direct Products

Theorem (8.6)**.** Let $\{N_i \mid i \in I\}$ be a family of normal subgroups of a group G. Suppose

- $G = \langle \bigcup_{i \in I} N_i \rangle$,
- for each $k \in I$, $N_k \cap \langle \bigcup_{i \neq k} N_i \rangle = \{e\}$.

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Proof. “$\Rightarrow$”: This is because the map $f : \prod_{i \in I}^{W} N_i \to G$ defined by $(e)_{i \in I} \mapsto e$ and $(a_i)_{i \in I} \mapsto \prod_{i \in I_{\neq e}} a_i$,

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Proof. "$\Leftarrow$": Since every element is a finite product of elements in $\bigcup_{i\in I} N_i$, we have $G = \langle \bigcup_{i\in I} N_i \rangle$. Moreover, using the uniqueness of the expression, we can see that for each $k \in I$, $N_k \cap \langle \bigcup_{i\neq k} N_i \rangle = \{e\}$. More precisely, any nonidentity element in N_k can not be a product of elements in $\bigcup_{i\neq k} N_i$.

Remark

Combining Definition (8.7) and the proof of Theorem (8.6), we get the following theorem:

Theorem (8.9). Let $\{N_i \mid i \in I\}$ be a family of normal subgroups of a group G. Then G is the internal weak direct product of the family $\{N_i \mid i \in I\}$ if and only if every nonidentity element of G is a unique product $a_{i_1} a_{i_2} \cdots a_{i_\ell}$ with $i_1, i_2, \ldots, i_\ell$ distinct elements of I and $e \neq a_{i_k} \in N_{i_k}$ for each $k = 1, 2, \ldots, \ell$.

Proof. "$\Leftarrow$": Since every element is a finite product of elements in $\bigcup_{i \in I} N_i$, we have $G = \langle \bigcup_{i \in I} N_i \rangle$. Moreover, using the uniqueness of the expression, we can see that for each $k \in I$, $N_k \cap \langle \bigcup_{i \neq k} N_i \rangle = \{e\}$. More precisely, any nonidentity element in N_k can not be a product of elements in $\bigcup_{i \neq k} N_i$. Hence G is the internal weak direct product of the family $\{N_i \mid i \in I\}$.

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Notation. We write $G = \prod_{i\in I}^{W} N_i$ to indicate that the group G is the internal direct product of the family of its subgroups $\{N_i \mid i \in I\}$.

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Proof. This can be checked directly.

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Consequently f is a monomorphism [resp. epimorphism] if and only if each f_i is.

Corollary (8.11)

Let $\{G_i \mid i \in I\}$ and $\{N_i \mid i \in I\}$ be families of groups such that N_i is a normal subgroup of G_i for each $i \in I$.

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(i) $\prod_{i\in I} N_i$ is a normal subgroup of $\prod_{i\in I} G_i$ and $\prod_{i\in I} G_i \big/ \prod_{i\in I} N_i \cong \prod_{i\in I} G_i/N_i$.

(ii) $\prod_{i\in I}^{W} N_i$ is a normal subgroup of $\prod_{i\in I}^{W} G_i$ and $\prod_{i\in I}^{W} G_i \big/ \prod_{i\in I}^{W} N_i \cong \prod_{i\in I}^{W} G_i/N_i$.

Proof. For each $i \in I$,

Corollary (8.11)

Let $\{G_i \mid i \in I\}$ and $\{N_i \mid i \in I\}$ be families of groups such that N_i is a normal subgroup of G_i for each $i \in I$.

(i) $\prod_{i\in I} N_i$ is a normal subgroup of $\prod_{i\in I} G_i$ and $\prod_{i\in I} G_i \big/ \prod_{i\in I} N_i \cong \prod_{i\in I} G_i/N_i$.

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Proof. For each $i \in I$, let $\pi_i : G_i \to G_i/N_i$ be the canonical epimorphism.

Corollary (8.11)

Let $\{G_i \mid i \in I\}$ and $\{N_i \mid i \in I\}$ be families of groups such that N_i is a normal subgroup of G_i for each $i \in I$.

(i) $\prod_{i\in I} N_i$ is a normal subgroup of $\prod_{i\in I} G_i$ and $\prod_{i\in I} G_i \diagup \prod_{i\in I} N_i \cong \prod_{i\in I} G_i/N_i$.

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Proof. For each $i \in I$, let $\pi_i : G_i \to G_i/N_i$ be the canonical epimorphism. By Theorem (8.10),

Corollary (8.11)

Let $\{G_i \mid i \in I\}$ and $\{N_i \mid i \in I\}$ be families of groups such that N_i is a normal subgroup of G_i for each $i \in I$.

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(ii) $\prod_{i\in I}^{W} N_i$ is a normal subgroup of $\prod_{i\in I}^{W} G_i$ and $\prod_{i\in I}^{W} G_i \big/ \prod_{i\in I}^{W} N_i \cong \prod_{i\in I}^{W} G_i/N_i$.

Proof. For each $i \in I$, let $\pi_i : G_i \to G_i/N_i$ be the canonical epimorphism. By Theorem (8.10), the map $\prod_{i\in I} \pi_i : \prod_{i\in I} G_i \to \prod_{i\in I} G_i/N_i$ is an epimorphism

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Proof. For each $i \in I$, let $\pi_i : G_i \to G_i/N_i$ be the canonical epimorphism. By Theorem (8.10), the map $\prod_{i\in I} \pi_i : \prod_{i\in I} G_i \to \prod_{i\in I} G_i/N_i$ is an epimorphism with kernel $\prod_{i\in I} N_i$.

Corollary (8.11)

Let $\{G_i \mid i \in I\}$ and $\{N_i \mid i \in I\}$ be families of groups such that N_i is a normal subgroup of G_i for each $i \in I$.

(i) $\prod_{i\in I} N_i$ is a normal subgroup of $\prod_{i\in I} G_i$ and $\prod_{i\in I} G_i \big/ \prod_{i\in I} N_i \cong \prod_{i\in I} G_i/N_i$.

(ii) $\prod^W_{i\in I} N_i$ is a normal subgroup of $\prod^W_{i\in I} G_i$ and $\prod^W_{i\in I} G_i \big/ \prod^W_{i\in I} N_i \cong \prod^W_{i\in I} G_i/N_i$.

Proof. For each $i \in I$, let $\pi_i : G_i \to G_i/N_i$ be the canonical epimorphism. By Theorem (8.10), the map $\prod_{i\in I} \pi_i : \prod_{i\in I} G_i \to \prod_{i\in I} G_i/N_i$ is an epimorphism with kernel $\prod_{i\in I} N_i$. Therefore $\prod_{i\in I} G_i \big/ \prod_{i\in I} N_i \cong \prod_{i\in I} G_i/N_i$ by the First Isomorphism Theorem,

Corollary (8.11)

Let $\{G_i \mid i \in I\}$ and $\{N_i \mid i \in I\}$ be families of groups such that N_i is a normal subgroup of G_i for each $i \in I$.

(i) $\prod_{i\in I} N_i$ is a normal subgroup of $\prod_{i\in I} G_i$ and $\prod_{i\in I} G_i \big/ \prod_{i\in I} N_i \cong \prod_{i\in I} G_i/N_i$.

(ii) $\prod_{i\in I}^W N_i$ is a normal subgroup of $\prod_{i\in I}^W G_i$ and $\prod_{i\in I}^W G_i \big/ \prod_{i\in I}^W N_i \cong \prod_{i\in I}^W G_i/N_i$.

Proof. For each $i \in I$, let $\pi_i : G_i \to G_i/N_i$ be the canonical epimorphism. By Theorem (8.10), the map $\prod_{i\in I} \pi_i : \prod_{i\in I} G_i \to \prod_{i\in I} G_i/N_i$ is an epimorphism with kernel $\prod_{i\in I} N_i$. Therefore $\prod_{i\in I} G_i \big/ \prod_{i\in I} N_i \cong \prod_{i\in I} G_i/N_i$ by the First Isomorphism Theorem, and this shows (i).

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Proof. For each $i \in I$, let $\pi_i : G_i \to G_i/N_i$ be the canonical epimorphism. By Theorem (8.10), the map $\prod_{i\in I} \pi_i : \prod_{i\in I} G_i \to \prod_{i\in I} G_i/N_i$ is an epimorphism with kernel $\prod_{i\in I} N_i$. Therefore $\prod_{i\in I} G_i \big/ \prod_{i\in I} N_i \cong \prod_{i\in I} G_i/N_i$ by the First Isomorphism Theorem, and this shows (i). Again by Theorem (8.10), we have $\prod_{i\in I} \pi_i \left(\prod_{i\in I}^{W} G_i\right) \subseteq \prod_{i\in I}^{W} G_i/N_i$.

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Let $\{G_i \mid i \in I\}$ and $\{N_i \mid i \in I\}$ be families of groups such that N_i is a normal subgroup of G_i for each $i \in I$.

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Proof. For each $i \in I$, let $\pi_i : G_i \to G_i/N_i$ be the canonical epimorphism. Hence we have a group homomorphism $\prod^W_{i\in I} \pi_i : \prod^W_{i\in I} G_i \to \prod^W_{i\in I} G_i/N_i$.

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Proof. For each $i \in I$, let $\pi_i : G_i \to G_i/N_i$ be the canonical epimorphism. Hence we have a group homomorphism $\prod_{i\in I}^{W} \pi_i : \prod_{i\in I}^{W} G_i \to \prod_{i\in I}^{W} G_i/N_i$. It can be checked directly that this map is indeed an epimorphism

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Proof. For each $i \in I$, let $\pi_i : G_i \to G_i/N_i$ be the canonical epimorphism. Hence we have a group homomorphism $\prod^W_{i\in I} \pi_i : \prod^W_{i\in I} G_i \to \prod^W_{i\in I} G_i/N_i$. It can be checked directly that this map is indeed an epimorphism with kernel $\prod^W_{i\in I} N_i$.

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Proof. For each $i \in I$, let $\pi_i : G_i \to G_i/N_i$ be the canonical epimorphism. Hence we have a group homomorphism $\prod_{i\in I}^{W} \pi_i : \prod_{i\in I}^{W} G_i \to \prod_{i\in I}^{W} G_i/N_i$. It can be checked directly that this map is indeed an epimorphism with kernel $\prod_{i\in I}^{W} N_i$. By the First Isomorphism Theorem, we prove (ii).

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Proof. For each $i \in I$, let $\pi_i : G_i \to G_i/N_i$ be the canonical epimorphism. Hence we have a group homomorphism $\prod_{i\in I}^{W} \pi_i : \prod_{i\in I}^{W} G_i \to \prod_{i\in I}^{W} G_i/N_i$. It can be checked directly that this map is indeed an epimorphism with kernel $\prod_{i\in I}^{W} N_i$. By the First Isomorphism Theorem, we prove (ii).

Exercise for Section I.8

2,3,4,5,12,14

Chapter II

THE STRUCTURE OF GROUPS

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Section II.1: Free Abelian Groups

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Section II.1: Free Abelian Groups

Remark. In this section, we will investigate free objects in the category of abelian groups. Through out this section, we will use *additive notation*. The following dictionary may be helpful converting what we have seen in Chapter one.

Multiplicative vs Additive

Multiplicative vs Additive

multiplicative notation

additive notation

Multiplicative vs Additive

multiplicative notation	additive notation
ab	$a + b$

Multiplicative vs Additive

multiplicative notation	additive notation
ab	$a+b$
a^{-1}	$-a$

Multiplicative vs Additive

multiplicative notation	additive notation
ab	$a+b$
a^{-1}	$-a$
e	0

Multiplicative vs Additive

multiplicative notation	additive notation
ab	$a+b$
a^{-1}	$-a$
e	0
a^n	na

Multiplicative vs Additive

multiplicative notation	additive notation
ab	$a+b$
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e	0
a^n	na
ab^{-1}	$a-b$

Multiplicative vs Additive

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ab	$a+b$
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HK	$H+K$

Multiplicative vs Additive

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HK	$H+K$
aH	$a+H$
$G\times H$	$G\oplus H$

Multiplicative vs Additive

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aH	$a+H$
$G \times H$	$G \oplus H$
$H \vee K$	$H+K$

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$G \times H$	$G \oplus H$
$H \vee K$	$H+K$
$\prod_{i\in I}^{W} G_i$	$\sum_{i\in I} G_i$

Multiplicative vs Additive

multiplicative notation	additive notation
ab	$a + b$
a^{-1}	$-a$
e	0
a^n	na
ab^{-1}	$a - b$
HK	$H + K$
aH	$a + H$
$G \times H$	$G \oplus H$
$H \vee K$	$H + K$
$\prod_{i\in I}^{W} G_i$	$\sum_{i\in I} G_i$
weak direct product	direct sum

Free Abelian Groups

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Definition. Let G be an abelian group and let $X \subseteq G$.

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- X is linearly independent

Free Abelian Groups

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Free Abelian Groups

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Definition. Let G be an abelian group and let $X \subseteq G$.

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- For every $z \in X$, $N_z \cap \langle \bigcup_{x \neq z} N_x \rangle = \{0\}$. This is because if for some $z \in X$, $N_z \cap \langle \bigcup_{x \neq z} N_x \rangle \neq \{0\}$, then there exist nonzero $n, n_1, \ldots, n_k \in \mathbb{Z}$ and distinct $x_1, \ldots, x_k \in X \setminus \{z\}$ such that $nz = n_1 x_1 + \cdots + n_k x_k$, but this contradicts the fact that X is a basis.

Proof of (i) $\Rightarrow$ (ii)

(i) F has a nonempty basis.

(ii) F is the (internal) direct sum of a family of infinite cyclic subgroups.

Proof. Let X be a nonempty basis of F, and for each $x \in X$, let $N_x = \langle x \rangle$. We first show that F is the internal direct sum of $\{N_x \mid x \in X\}$. Hence F is the internal direct sum of $\{N_x \mid x \in X\}$.

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Proof of (i) $\Rightarrow$ (ii)

(i) F has a nonempty basis.

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Proof. Let X be a nonempty basis of F, and for each $x \in X$, let $N_x = \langle x \rangle$. We first show that F is the internal direct sum of $\{N_x \mid x \in X\}$. Hence F is the internal direct sum of $\{N_x \mid x \in X\}$. Moreover, because X is a basis of F, for each $x \in X$, $nx = 0 \implies n = 0$ and so x has infinite order, i.e., $N_x = \langle x \rangle$ is infinite cyclic. Therefore, F is the internal direct sum of $\{N_x \mid x \in X\}$,

Proof of (i) $\Rightarrow$ (ii)

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Proof. Let X be a nonempty basis of F, and for each $x \in X$, let $N_x = \langle x \rangle$. We first show that F is the internal direct sum of $\{N_x \mid x \in X\}$. Hence F is the internal direct sum of $\{N_x \mid x \in X\}$. Moreover, because X is a basis of F, for each $x \in X$, $nx = 0 \implies n = 0$ and so x has infinite order, i.e., $N_x = \langle x \rangle$ is infinite cyclic. Therefore, F is the internal direct sum of $\{N_x \mid x \in X\}$, which is a family of infinite cyclic subgroups.

Theorem (1.1)

The following conditions on an abelian group F are equivalent:

(i) F has a nonempty basis.

(ii) F is the (internal) direct sum of a family of infinite cyclic subgroups.

(iii) F is (isomorphic to) a direct sum of copies of the additive groups $\mathbb{Z}$ of integers, i.e., $F \cong \sum_{i \in I} \mathbb{Z}$ for some index set I.

(iv) F is a free object in the category of abelian groups, i.e., there exists a nonempty set X and a function $\iota : X \to F$ with the following property: given an abelian group G and function $f : X \to G$, there exists a unique homomorphism of groups $\bar{f} : F \to G$ such that $\bar{f}\iota = f$.

Proof. We have shown (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) $\Rightarrow$ (iv) $\Rightarrow$ (iii).

Proof of (ii) $\Rightarrow$ (iii)

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Proof.

Proof of (ii) $\Rightarrow$ (iii)

(ii) F is the (internal) direct sum of a family of infinite cyclic subgroups.

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Proof. Suppose F is the internal direct sum of $\{N_i \mid i \in I\}$ where N_i is infinite cyclic.

Proof of (ii) $\Rightarrow$ (iii)

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Proof. Suppose F is the internal direct sum of $\{N_i \mid i \in I\}$ where N_i is infinite cyclic. Then by Theorem (I.8.6), $F \cong \sum_{i \in I} N_i$.

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Proof. Suppose F is the internal direct sum of $\{N_i \mid i \in I\}$ where N_i is infinite cyclic. Then by Theorem (I.8.6), $F \cong \sum_{i \in I} N_i$. Moreover, for every $i \in I$, since N_i is infinite cyclic,

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Proof. Suppose F is the internal direct sum of $\{N_i \mid i \in I\}$ where N_i is infinite cyclic. Then by Theorem (I.8.6), $F \cong \sum_{i\in I} N_i$. Moreover, for every $i \in I$, since N_i is infinite cyclic, $N_i \cong \mathbb{Z}$. Hence, we have $F \cong \sum_{i\in I} N_i$

Proof of (ii) $\Rightarrow$ (iii)

(ii) F is the (internal) direct sum of a family of infinite cyclic subgroups.

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Proof. Suppose F is the internal direct sum of $\{N_i \mid i \in I\}$ where N_i is infinite cyclic. Then by Theorem (I.8.6), $F \cong \sum_{i \in I} N_i$. Moreover, for every $i \in I$, since N_i is infinite cyclic, $N_i \cong \mathbb{Z}$. Hence, we have $F \cong \sum_{i \in I} N_i \cong \sum_{i \in I} \mathbb{Z}$.

Theorem (1.1)

The following conditions on an abelian group F are equivalent:

(i) F has a nonempty basis.

(ii) F is the (internal) direct sum of a family of infinite cyclic subgroups.

(iii) F is (isomorphic to) a direct sum of copies of the additive groups $\mathbb{Z}$ of integers, i.e., $F \cong \sum_{i \in I} \mathbb{Z}$ for some index set I.

(iv) F is a free object in the category of abelian groups, i.e., there exists a nonempty set X and a function $\iota : X \to F$ with the following property: given an abelian group G and function $f : X \to G$, there exists a unique homomorphism of groups $\bar{f} : F \to G$ such that $\bar{f}\iota = f$.

Proof. We have shown (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) $\Rightarrow$ (iv) $\Rightarrow$ (iii).

Proof of (iii) $\Rightarrow$ (i)

(i) F has a nonempty basis.

(iii) F is (isomorphic to) a direct sum of copies of the additive groups $\mathbb{Z}$ of integers, i.e., $F \cong \sum_{i \in I} \mathbb{Z}$ for some index set I.

Proof of (iii) $\Rightarrow$ (i)

(i) F has a nonempty basis.

(iii) F is (isomorphic to) a direct sum of copies of the additive groups $\mathbb{Z}$ of integers, i.e., $F \cong \sum_{i \in I} \mathbb{Z}$ for some index set I.

Proof.

Proof of (iii) $\Rightarrow$ (i)

(i) F has a nonempty basis.

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Proof. Suppose $\varphi : \sum_{i\in I} \mathbb{Z} \to F$ is a group isomorphism.

Proof of (iii) $\Rightarrow$ (i)

(i) F has a nonempty basis.

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Proof. Suppose $\varphi : \sum_{i \in I} \mathbb{Z} \to F$ is a group isomorphism. Let $\mathbf{e}_k = (a_i)_{i \in I} \in \sum_{i \in I} \mathbb{Z}$ be such that $\begin{cases} a_k = 1 \\ a_i = 0 \quad \forall i \neq k. \end{cases}$

Proof of (iii) $\Rightarrow$ (i)

(i) F has a nonempty basis.

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Using the fact that $\{\mathbf{e}_i \mid i \in I\}$ is a basis of $\sum_{i\in I} \mathbb{Z}$

Proof of (iii) $\Rightarrow$ (i)

(i) F has a nonempty basis.

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Proof. Suppose $\varphi : \sum_{i \in I} \mathbb{Z} \to F$ is a group isomorphism. Let $\mathbf{e}_k = (a_i)_{i \in I} \in \sum_{i \in I} \mathbb{Z}$ be such that $\begin{cases} a_k = 1 \\ a_i = 0 \quad \forall i \neq k. \end{cases}$

Using the fact that $\{\mathbf{e}_i \mid i \in I\}$ is a basis of $\sum_{i \in I} \mathbb{Z}$ and φ is a group isomorphism,

Proof of (iii) $\Rightarrow$ (i)

(i) F has a nonempty basis.

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Proof. Suppose $\varphi : \sum_{i\in I} \mathbb{Z} \to F$ is a group isomorphism. Let $\mathbf{e}_k = (a_i)_{i\in I} \in \sum_{i\in I} \mathbb{Z}$ be such that $\begin{cases} a_k = 1 \\ a_i = 0 \quad \forall i \neq k. \end{cases}$

Using the fact that $\{\mathbf{e}_i \mid i \in I\}$ is a basis of $\sum_{i\in I} \mathbb{Z}$ and φ is a group isomorphism, we will show that $\{\varphi(\mathbf{e}_i) \mid i \in I\}$ is a basis of F.

Proof of (iii) $\Rightarrow$ (i)

(i) F has a nonempty basis.

(iii) F is (isomorphic to) a direct sum of copies of the additive groups $\mathbb{Z}$ of integers, i.e., $F \cong \sum_{i \in I} \mathbb{Z}$ for some index set I.

Proof. Suppose $\varphi : \sum_{i \in I} \mathbb{Z} \to F$ is a group isomorphism. Let $\mathbf{e}_k = (a_i)_{i \in I} \in \sum_{i \in I} \mathbb{Z}$ be such that $\begin{cases} a_k = 1 \\ a_i = 0 \quad \forall i \neq k. \end{cases}$

Let $a \in F$ be a nonzero element.

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Let $a \in F$ be a nonzero element. Since φ is onto,

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Let $a \in F$ be a nonzero element. Since φ is onto, $\varphi\big((n_i)_{i\in I}\big) = a$, for some $(n_i)_{i\in I} \in \sum_{i\in I} \mathbb{Z}$.

Proof of (iii) $\Rightarrow$ (i)

(i) F has a nonempty basis.

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(i) F has a nonempty basis.

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Let $a \in F$ be a nonzero element. Since φ is onto, $\varphi\big((n_i)_{i\in I}\big) = a$, for some $(n_i)_{i\in I} \in \sum_{i\in I} \mathbb{Z}$. Since $(n_i)_{i\in I} = \sum_{i\in I_{\neq 0}} n_i \mathbf{e}_i$, where $I_{\neq 0} = \{i \in I \mid n_i \neq 0\}$,

$a = \varphi\big((n_i)_{i\in I}\big)$

Proof of (iii) $\Rightarrow$ (i)

(i) F has a nonempty basis.

(iii) F is (isomorphic to) a direct sum of copies of the additive groups $\mathbb{Z}$ of integers, i.e., $F \cong \sum_{i\in I} \mathbb{Z}$ for some index set I.

Proof. Suppose $\varphi : \sum_{i\in I} \mathbb{Z} \to F$ is a group isomorphism. Let $\mathbf{e}_k = (a_i)_{i\in I} \in \sum_{i\in I} \mathbb{Z}$ be such that $\begin{cases} a_k = 1 \\ a_i = 0 \quad \forall i \neq k. \end{cases}$

Let $a \in F$ be a nonzero element. Since φ is onto, $\varphi\big((n_i)_{i\in I}\big) = a$, for some $(n_i)_{i\in I} \in \sum_{i\in I} \mathbb{Z}$. Since $(n_i)_{i\in I} = \sum_{i\in I_{\neq 0}} n_i\mathbf{e}_i$, where $I_{\neq 0} = \{i \in I \mid n_i \neq 0\}$,
$a = \varphi\big((n_i)_{i\in I}\big) = \varphi\big(\sum_{i\in I_{\neq 0}} n_i\mathbf{e}_i\big)$

Proof of (iii) $\Rightarrow$ (i)

(i) F has a nonempty basis.

(iii) F is (isomorphic to) a direct sum of copies of the additive groups $\mathbb{Z}$ of integers, i.e., $F \cong \sum_{i \in I} \mathbb{Z}$ for some index set I.

Proof. Suppose $\varphi : \sum_{i \in I} \mathbb{Z} \to F$ is a group isomorphism. Let $\mathbf{e}_k = (a_i)_{i \in I} \in \sum_{i \in I} \mathbb{Z}$ be such that $\begin{cases} a_k = 1 \\ a_i = 0 \quad \forall i \neq k. \end{cases}$

Let $a \in F$ be a nonzero element. Since φ is onto, $\varphi\big((n_i)_{i \in I}\big) = a$, for some $(n_i)_{i \in I} \in \sum_{i \in I} \mathbb{Z}$. Since $(n_i)_{i \in I} = \sum_{i \in I_{\neq 0}} n_i \mathbf{e}_i$, where $I_{\neq 0} = \{i \in I \mid n_i \neq 0\}$, $a = \varphi\big((n_i)_{i \in I}\big) = \varphi\big(\sum_{i \in I_{\neq 0}} n_i \mathbf{e}_i\big) = \sum_{i \in I_{\neq 0}} n_i \varphi(\mathbf{e}_i)$.

Proof of (iii) $\Rightarrow$ (i)

(i) F has a nonempty basis.

(iii) F is (isomorphic to) a direct sum of copies of the additive groups $\mathbb{Z}$ of integers, i.e., $F \cong \sum_{i \in I} \mathbb{Z}$ for some index set I.

Proof. Suppose $\varphi : \sum_{i \in I} \mathbb{Z} \to F$ is a group isomorphism. Let $\mathbf{e}_k = (a_i)_{i \in I} \in \sum_{i \in I} \mathbb{Z}$ be such that $\begin{cases} a_k = 1 \\ a_i = 0 \quad \forall i \neq k. \end{cases}$

Let $a \in F$ be a nonzero element. Since φ is onto, $\varphi\big((n_i)_{i \in I}\big) = a$, for some $(n_i)_{i \in I} \in \sum_{i \in I} \mathbb{Z}$. Since $(n_i)_{i \in I} = \sum_{i \in I_{\neq 0}} n_i \mathbf{e}_i$, where $I_{\neq 0} = \{i \in I \mid n_i \neq 0\}$, $a = \varphi\big((n_i)_{i \in I}\big) = \varphi\big(\sum_{i \in I_{\neq 0}} n_i \mathbf{e}_i\big) = \sum_{i \in I_{\neq 0}} n_i \varphi(\mathbf{e}_i)$. Hence, $\{\varphi(\mathbf{e}_i) \mid i \in I\}$ generates F.

Proof of (iii) $\Rightarrow$ (i)

(i) F has a nonempty basis.

(iii) F is (isomorphic to) a direct sum of copies of the additive groups $\mathbb{Z}$ of integers, i.e., $F \cong \sum_{i\in I} \mathbb{Z}$ for some index set I.

Proof. Suppose $\varphi : \sum_{i\in I} \mathbb{Z} \to F$ is a group isomorphism. Let $\mathbf{e}_k = (a_i)_{i\in I} \in \sum_{i\in I} \mathbb{Z}$ be such that $\begin{cases} a_k = 1 \\ a_i = 0 \quad \forall i \neq k. \end{cases}$

Hence, $\{\varphi(\mathbf{e}_i) \mid i \in I\}$ generates F. Suppose

$$n_{i_1}\varphi(\mathbf{e}_{i_1}) + n_{i_2}\varphi(\mathbf{e}_{i_2}) + \cdots + n_{i_k}\varphi(\mathbf{e}_{i_k}) = 0$$

Proof of (iii) $\Rightarrow$ (i)

(i) F has a nonempty basis.

(iii) F is (isomorphic to) a direct sum of copies of the additive groups $\mathbb{Z}$ of integers, i.e., $F \cong \sum_{i \in I} \mathbb{Z}$ for some index set I.

Proof. Suppose $\varphi : \sum_{i \in I} \mathbb{Z} \to F$ is a group isomorphism. Let $\mathbf{e}_k = (a_i)_{i \in I} \in \sum_{i \in I} \mathbb{Z}$ be such that $\begin{cases} a_k = 1 \\ a_i = 0 \quad \forall i \neq k. \end{cases}$

Hence, $\{\varphi(\mathbf{e}_i) \mid i \in I\}$ generates F. Suppose $n_{i_1}\varphi(\mathbf{e}_{i_1}) + n_{i_2}\varphi(\mathbf{e}_{i_2}) + \cdots + n_{i_k}\varphi(\mathbf{e}_{i_k}) = 0$ where $\varphi(\mathbf{e}_{i_1}), \varphi(\mathbf{e}_{i_1}), \ldots, \varphi(\mathbf{e}_{i_1})$ are distinct

Proof of (iii) $\Rightarrow$ (i)

(i) F has a nonempty basis.

(iii) F is (isomorphic to) a direct sum of copies of the additive groups $\mathbb{Z}$ of integers, i.e., $F \cong \sum_{i\in I}\mathbb{Z}$ for some index set I.

Proof. Suppose $\varphi : \sum_{i\in I}\mathbb{Z} \to F$ is a group isomorphism. Let $\mathbf{e}_k = (a_i)_{i\in I} \in \sum_{i\in I}\mathbb{Z}$ be such that $\begin{cases} a_k = 1 \\ a_i = 0 \quad \forall i \neq k. \end{cases}$

Hence, $\{\varphi(\mathbf{e}_i) \mid i \in I\}$ generates F. Suppose $n_{i_1}\varphi(\mathbf{e}_{i_1}) + n_{i_2}\varphi(\mathbf{e}_{i_2}) + \cdots + n_{i_k}\varphi(\mathbf{e}_{i_k}) = 0$ where $\varphi(\mathbf{e}_{i_1}), \varphi(\mathbf{e}_{i_1}), \ldots, \varphi(\mathbf{e}_{i_1})$ are distinct and $n_{i_1}, n_{i_2}, \ldots, n_{i_k} \in \mathbb{Z}$.

Proof of (iii) $\Rightarrow$ (i)

(i) F has a nonempty basis.

(iii) F is (isomorphic to) a direct sum of copies of the additive groups $\mathbb{Z}$ of integers, i.e., $F \cong \sum_{i\in I} \mathbb{Z}$ for some index set I.

Proof. Suppose $\varphi : \sum_{i\in I} \mathbb{Z} \to F$ is a group isomorphism. Let $\mathbf{e}_k = (a_i)_{i\in I} \in \sum_{i\in I} \mathbb{Z}$ be such that $\begin{cases} a_k = 1 \\ a_i = 0 \quad \forall i \neq k. \end{cases}$

Hence, $\{\varphi(\mathbf{e}_i) \mid i \in I\}$ generates F. Suppose $n_{i_1}\varphi(\mathbf{e}_{i_1}) + n_{i_2}\varphi(\mathbf{e}_{i_2}) + \cdots + n_{i_k}\varphi(\mathbf{e}_{i_k}) = 0$ where $\varphi(\mathbf{e}_{i_1}), \varphi(\mathbf{e}_{i_1}), \ldots, \varphi(\mathbf{e}_{i_1})$ are distinct and $n_{i_1}, n_{i_2}, \ldots, n_{i_k} \in \mathbb{Z}$. Then $\varphi(n_{i_1}\mathbf{e}_{i_1} + n_{i_2}\mathbf{e}_{i_2} + \cdots + n_{i_k}\mathbf{e}_{i_k}) = 0$.

Proof of (iii) $\Rightarrow$ (i)

(i) F has a nonempty basis.

(iii) F is (isomorphic to) a direct sum of copies of the additive groups $\mathbb{Z}$ of integers, i.e., $F \cong \sum_{i\in I} \mathbb{Z}$ for some index set I.

Proof. Suppose $\varphi : \sum_{i\in I} \mathbb{Z} \to F$ is a group isomorphism. Let $\mathbf{e}_k = (a_i)_{i\in I} \in \sum_{i\in I} \mathbb{Z}$ be such that $\begin{cases} a_k = 1 \\ a_i = 0 \quad \forall i \neq k. \end{cases}$

Hence, $\{\varphi(\mathbf{e}_i) \mid i \in I\}$ generates F. Suppose $n_{i_1}\varphi(\mathbf{e}_{i_1}) + n_{i_2}\varphi(\mathbf{e}_{i_2}) + \cdots + n_{i_k}\varphi(\mathbf{e}_{i_k}) = 0$ where $\varphi(\mathbf{e}_{i_1}), \varphi(\mathbf{e}_{i_1}), \ldots, \varphi(\mathbf{e}_{i_1})$ are distinct and $n_{i_1}, n_{i_2}, \ldots, n_{i_k} \in \mathbb{Z}$. Then $\varphi(n_{i_1}\mathbf{e}_{i_1} + n_{i_2}\mathbf{e}_{i_2} + \cdots + n_{i_k}\mathbf{e}_{i_k}) = 0$. Since φ is one-to-one,

Proof of (iii) $\Rightarrow$ (i)

(i) F has a nonempty basis.

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Proof of (iii) $\Rightarrow$ (i)

(i) F has a nonempty basis.

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Proof of (iii) $\Rightarrow$ (i)

(i) F has a nonempty basis.

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(i) F has a nonempty basis.

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Hence, $\{\varphi(\mathbf{e}_i) \mid i \in I\}$ generates F. Suppose $n_{i_1}\varphi(\mathbf{e}_{i_1}) + n_{i_2}\varphi(\mathbf{e}_{i_2}) + \cdots + n_{i_k}\varphi(\mathbf{e}_{i_k}) = 0$ where $\varphi(\mathbf{e}_{i_1}), \varphi(\mathbf{e}_{i_1}), \ldots, \varphi(\mathbf{e}_{i_1})$ are distinct and $n_{i_1}, n_{i_2}, \ldots, n_{i_k} \in \mathbb{Z}$. Then $\varphi(n_{i_1}\mathbf{e}_{i_1} + n_{i_2}\mathbf{e}_{i_2} + \cdots + n_{i_k}\mathbf{e}_{i_k}) = 0$. Since φ is one-to-one, $n_{i_1}\mathbf{e}_{i_1} + n_{i_2}\mathbf{e}_{i_2} + \cdots + n_{i_k}\mathbf{e}_{i_k} = 0$. Because $\{e_i \mid i \in I\}$ is linearly independent, we get $n_{i_1} = \cdots = n_{i_k} = 0$. Therefore, $\{\varphi(\mathbf{e}_i) \mid i \in I\}$ is linearly independent.

Proof of (iii) $\Rightarrow$ (i)

(i) F has a nonempty basis.

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Hence, $\{\varphi(\mathbf{e}_i) \mid i \in I\}$ generates F. Therefore, $\{\varphi(\mathbf{e}_i) \mid i \in I\}$ is linearly independent. Hence, $\{\varphi(\mathbf{e}_i) \mid i \in I\}$ is a basis of F.

Theorem (1.1)

The following conditions on an abelian group F are equivalent:

(i) F has a nonempty basis.

(ii) F is the (internal) direct sum of a family of infinite cyclic subgroups.

(iii) F is (isomorphic to) a direct sum of copies of the additive groups $\mathbb{Z}$ of integers, i.e., $F \cong \sum_{i \in I} \mathbb{Z}$ for some index set I.

(iv) F is a free object in the category of abelian groups, i.e., there exists a nonempty set X and a function $\iota : X \to F$ with the following property: given an abelian group G and function $f : X \to G$, there exists a unique homomorphism of groups $\bar{f} : F \to G$ such that $\bar{f}\iota = f$.

Proof. We have shown (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii)$\Rightarrow$ (i) $\Rightarrow$ (iv) $\Rightarrow$ (iii).