Modern Algebra I

Lecture 6

Jung-Chen Liu

liujc@math.ntnu.edu.tw

2009, Fall

Today, we will continue the section

Section I.7: Categories: Products, Coproducts, and Free Objects

Today, we will continue the section

Section I.7: Categories: Products, Coproducts, and Free Objects

and start to cover the section

Section I.8: Direct Products and Direct Sums

Chapter I

Section I.7: Categories: Products, Coproducts, and Free Objects

Chapter I

Section I.7: Categories: Products, Coproducts, and Free Objects

We first review some of the material that we covered last week.

Cartesian Products of Sets

Definition. Let $\{A_i \mid i \in I\}$ be a family of sets indexed by a nonempty set I.

Cartesian Products of Sets

Definition. Let $\{A_i \mid i \in I\}$ be a family of sets indexed by a nonempty set I.

- The Cartesian product $\prod_{i \in I} A_i$ of the sets A_i

Cartesian Products of Sets

Definition. Let $\{A_i \mid i \in I\}$ be a family of sets indexed by a nonempty set I.

- The Cartesian product $\prod_{i \in I} A_i$ of the sets A_i is the set of all functions $f : I \to \bigcup_{i \in I} A_i$

Cartesian Products of Sets

Definition. Let $\{A_i \mid i \in I\}$ be a family of sets indexed by a nonempty set I.

- The Cartesian product $\prod_{i \in I} A_i$ of the sets A_i is the set of all functions $f : I \to \bigcup_{i \in I} A_i$ such that $f(i) \in A_i$ for all $i \in I$.

Cartesian Products of Sets

Definition. Let $\{A_i \mid i \in I\}$ be a family of sets indexed by a nonempty set I.

- The Cartesian product $\prod_{i \in I} A_i$ of the sets A_i is the set of all functions $f : I \to \bigcup_{i \in I} A_i$ such that $f(i) \in A_i$ for all $i \in I$.
- We write the element $f : I \to \bigcup_{i \in I} A_i$ of $\prod_{i \in I} A_i$ as $(a_i)_{i \in I}$,

Cartesian Products of Sets

Definition. Let $\{A_i \mid i \in I\}$ be a family of sets indexed by a nonempty set I.

- The Cartesian product $\prod_{i \in I} A_i$ of the sets A_i is the set of all functions $f : I \to \bigcup_{i \in I} A_i$ such that $f(i) \in A_i$ for all $i \in I$.
- We write the element $f : I \to \bigcup_{i \in I} A_i$ of $\prod_{i \in I} A_i$ as $(a_i)_{i \in I}$, where $a_i = f(i)$ for all $i \in I$.

Cartesian Products of Sets

Definition. Let $\{A_i \mid i \in I\}$ be a family of sets indexed by a nonempty set I.

- The Cartesian product $\prod_{i\in I} A_i$ of the sets A_i is the set of all functions $f : I \to \bigcup_{i\in I} A_i$ such that $f(i) \in A_i$ for all $i \in I$.
- We write the element $f : I \to \bigcup_{i\in I} A_i$ of $\prod_{i\in I} A_i$ as $(a_i)_{i\in I}$, where $a_i = f(i)$ for all $i \in I$.
- For each $k \in I$,

Cartesian Products of Sets

Definition. Let $\{A_i \mid i \in I\}$ be a family of sets indexed by a nonempty set I.

- The Cartesian product $\prod_{i \in I} A_i$ of the sets A_i is the set of all functions $f : I \to \bigcup_{i \in I} A_i$ such that $f(i) \in A_i$ for all $i \in I$.
- We write the element $f : I \to \bigcup_{i \in I} A_i$ of $\prod_{i \in I} A_i$ as $(a_i)_{i \in I}$, where $a_i = f(i)$ for all $i \in I$.
- For each $k \in I$, the map $\pi_k : \prod_{i \in I} A_i \to A_k$ defined by $f \mapsto f(k)$,

Cartesian Products of Sets

Definition. Let $\{A_i \mid i \in I\}$ be a family of sets indexed by a nonempty set I.

- The Cartesian product $\prod_{i\in I} A_i$ of the sets A_i is the set of all functions $f : I \to \bigcup_{i\in I} A_i$ such that $f(i) \in A_i$ for all $i \in I$.
- We write the element $f : I \to \bigcup_{i\in I} A_i$ of $\prod_{i\in I} A_i$ as $(a_i)_{i\in I}$, where $a_i = f(i)$ for all $i \in I$.
- For each $k \in I$, the map $\pi_k : \prod_{i\in I} A_i \to A_k$ defined by $f \mapsto f(k)$, i.e., $(a_i)_{i\in I} \mapsto a_k$,

Cartesian Products of Sets

Definition. Let $\{A_i \mid i \in I\}$ be a family of sets indexed by a nonempty set I.

- The Cartesian product $\prod_{i\in I} A_i$ of the sets A_i is the set of all functions $f : I \to \bigcup_{i\in I} A_i$ such that $f(i) \in A_i$ for all $i \in I$.
- We write the element $f : I \to \bigcup_{i\in I} A_i$ of $\prod_{i\in I} A_i$ as $(a_i)_{i\in I}$, where $a_i = f(i)$ for all $i \in I$.
- For each $k \in I$, the map $\pi_k : \prod_{i\in I} A_i \to A_k$ defined by $f \mapsto f(k)$, i.e., $(a_i)_{i\in I} \mapsto a_k$, is called the (canonical) projection of the product $\prod_{i\in I} A_i$

Cartesian Products of Sets

Definition. Let $\{A_i \mid i \in I\}$ be a family of sets indexed by a nonempty set I.

- The Cartesian product $\prod_{i \in I} A_i$ of the sets A_i is the set of all functions $f : I \to \bigcup_{i \in I} A_i$ such that $f(i) \in A_i$ for all $i \in I$.
- We write the element $f : I \to \bigcup_{i \in I} A_i$ of $\prod_{i \in I} A_i$ as $(a_i)_{i \in I}$, where $a_i = f(i)$ for all $i \in I$.
- For each $k \in I$, the map $\pi_k : \prod_{i \in I} A_i \to A_k$ defined by $f \mapsto f(k)$, i.e., $(a_i)_{i \in I} \mapsto a_k$, is called the (canonical) projection of the product $\prod_{i \in I} A_i$ onto its k-th component

Cartesian Products of Sets

Definition. Let $\{A_i \mid i \in I\}$ be a family of sets indexed by a nonempty set I.

- The Cartesian product $\prod_{i \in I} A_i$ of the sets A_i is the set of all functions $f : I \to \bigcup_{i \in I} A_i$ such that $f(i) \in A_i$ for all $i \in I$.
- We write the element $f : I \to \bigcup_{i \in I} A_i$ of $\prod_{i \in I} A_i$ as $(a_i)_{i \in I}$, where $a_i = f(i)$ for all $i \in I$.
- For each $k \in I$, the map $\pi_k : \prod_{i \in I} A_i \to A_k$ defined by $f \mapsto f(k)$, i.e., $(a_i)_{i \in I} \mapsto a_k$, is called the (canonical) projection of the product $\prod_{i \in I} A_i$ onto its k-th component (or k-th factor).

Theorem (Introduction, 5.2)

Let $\{A_i \mid i \in I\}$ be a family of (nonempty) sets indexed by a nonempty set I.

Theorem (Introduction, 5.2)

Let $\{A_i \mid i \in I\}$ be a family of (nonempty) sets indexed by a nonempty set I. Then there exists a set D,

Theorem (Introduction, 5.2)

Let $\{A_i \mid i \in I\}$ be a family of (nonempty) sets indexed by a nonempty set I. Then there exists a set D, together with a family of maps $\{\pi_i : D \to A_i \mid i \in I\}$

Theorem (Introduction, 5.2)

Let $\{A_i \mid i \in I\}$ be a family of (nonempty) sets indexed by a nonempty set I. Then there exists a set D, together with a family of maps $\{\pi_i : D \to A_i \mid i \in I\}$ satisfying the following property:

Theorem (Introduction, 5.2)

Let $\{A_i \mid i \in I\}$ be a family of (nonempty) sets indexed by a nonempty set I. Then there exists a set D, together with a family of maps $\{\pi_i : D \to A_i \mid i \in I\}$ satisfying the following property: for any set C and family of maps $\{\varphi_i : C \to A_i \mid i \in I\}$,

Theorem (Introduction, 5.2)

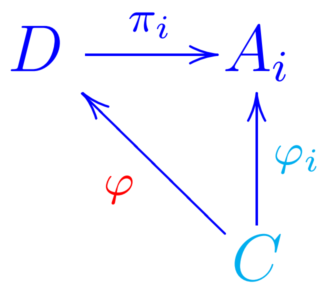
Let $\{A_i \mid i \in I\}$ be a family of (nonempty) sets indexed by a nonempty set I. Then there exists a set D, together with a family of maps $\{\pi_i : D \to A_i \mid i \in I\}$ satisfying the following property: for any set C and family of maps $\{\varphi_i : C \to A_i \mid i \in I\}$, there exists a unique map $\varphi : C \to D$

Theorem (Introduction, 5.2)

Let $\{A_i \mid i \in I\}$ be a family of (nonempty) sets indexed by a nonempty set I. Then there exists a set D, together with a family of maps $\{\pi_i : D \to A_i \mid i \in I\}$ satisfying the following property: for any set C and family of maps $\{\varphi_i : C \to A_i \mid i \in I\}$, there exists a unique map $\varphi : C \to D$ such that the diagram

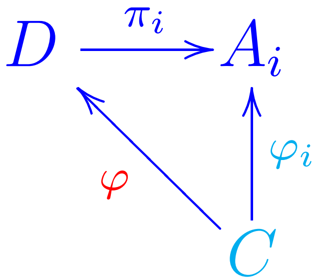
Theorem (Introduction, 5.2)

Let $\{A_i \mid i \in I\}$ be a family of (nonempty) sets indexed by a nonempty set I. Then there exists a set D, together with a family of maps $\{\pi_i : D \to A_i \mid i \in I\}$ satisfying the following property: for any set C and family of maps $\{\varphi_i : C \to A_i \mid i \in I\}$, there exists a unique map $\varphi : C \to D$ such that the diagram



Theorem (Introduction, 5.2)

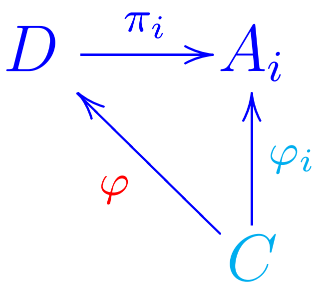
Let $\{A_i \mid i \in I\}$ be a family of (nonempty) sets indexed by a nonempty set I. Then there exists a set D, together with a family of maps $\{\pi_i : D \to A_i \mid i \in I\}$ satisfying the following property: for any set C and family of maps $\{\varphi_i : C \to A_i \mid i \in I\}$, there exists a unique map $\varphi : C \to D$ such that the diagram



commutes for all $i \in I$.

Theorem (Introduction, 5.2)

Let $\{A_i \mid i \in I\}$ be a family of (nonempty) sets indexed by a nonempty set I. Then there exists a set D, together with a family of maps $\{\pi_i : D \to A_i \mid i \in I\}$ satisfying the following property: for any set C and family of maps $\{\varphi_i : C \to A_i \mid i \in I\}$, there exists a unique map $\varphi : C \to D$ such that the diagram

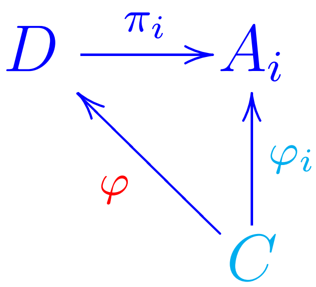


commutes for all $i \in I$.

Furthermore, D is uniquely determined up to a bijection.

Theorem (Introduction, 5.2)

Let $\{A_i \mid i \in I\}$ be a family of (nonempty) sets indexed by a nonempty set I. Then there exists a set D, together with a family of maps $\{\pi_i : D \to A_i \mid i \in I\}$ satisfying the following property: for any set C and family of maps $\{\varphi_i : C \to A_i \mid i \in I\}$, there exists a unique map $\varphi : C \to D$ such that the diagram



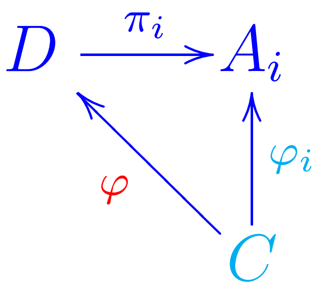
commutes for all $i \in I$.

Furthermore, D is uniquely determined up to a bijection.

In fact, we proved that the Cartesian product $\prod_{i \in I} A_i$

Theorem (Introduction, 5.2)

Let $\{A_i \mid i \in I\}$ be a family of (nonempty) sets indexed by a nonempty set I. Then there exists a set D, together with a family of maps $\{\pi_i : D \to A_i \mid i \in I\}$ satisfying the following property: for any set C and family of maps $\{\varphi_i : C \to A_i \mid i \in I\}$, there exists a unique map $\varphi : C \to D$ such that the diagram



commutes for all $i \in I$.

Furthermore, D is uniquely determined up to a bijection.

In fact, we proved that the Cartesian product $\prod_{i \in I} A_i$ together with the family of canonical projections $\{\pi_i : \prod_{i \in I} A_i \to A_i \mid i \in I\}$ satisfies the property for D.

Products in a Category

Products in a Category

Definition (7.2)**.** Let $\mathcal{C}$ be a category

Products in a Category

Definition (7.2)**.** Let $\mathcal{C}$ be a category and let $\{A_i \mid i \in I\}$ be a family of objects of $\mathcal{C}$.

Products in a Category

Definition (7.2)**.** Let $\mathcal{C}$ be a category and let $\{A_i \mid i \in I\}$ be a family of objects of $\mathcal{C}$. A product for the family $\{A_i \mid i \in I\}$

Products in a Category

Definition (7.2)**.** Let $\mathcal{C}$ be a category and let $\{A_i \mid i \in I\}$ be a family of objects of $\mathcal{C}$. A product for the family $\{A_i \mid i \in I\}$ is an object P of $\mathcal{C}$

Products in a Category

Definition (7.2)**.** Let $\mathcal{C}$ be a category and let $\{A_i \mid i \in I\}$ be a family of objects of $\mathcal{C}$. A product for the family $\{A_i \mid i \in I\}$ is an object P of $\mathcal{C}$ together with a family of morphisms $\{\pi_i : P \longrightarrow A_i \mid i \in I\}$

Products in a Category

Definition (7.2). Let $\mathcal{C}$ be a category and let $\{A_i \mid i \in I\}$ be a family of objects of $\mathcal{C}$. A product for the family $\{A_i \mid i \in I\}$ is an object P of $\mathcal{C}$ together with a family of morphisms $\{\pi_i : P \to A_i \mid i \in I\}$ such that for any object B

Products in a Category

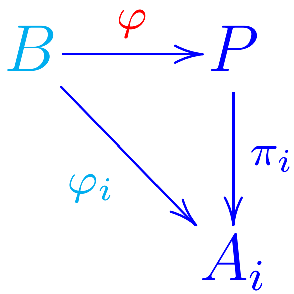
Definition (7.2)**.** Let $\mathcal{C}$ be a category and let $\{A_i \mid i \in I\}$ be a family of objects of $\mathcal{C}$. A product for the family $\{A_i \mid i \in I\}$ is an object P of $\mathcal{C}$ together with a family of morphisms $\{\pi_i : P \to A_i \mid i \in I\}$ such that for any object B and family of morphisms $\{\varphi_i : B \to A_i \mid i \in I\}$ in $\mathcal{C}$,

Products in a Category

Definition (7.2). Let $\mathcal{C}$ be a category and let $\{A_i \mid i \in I\}$ be a family of objects of $\mathcal{C}$. A product for the family $\{A_i \mid i \in I\}$ is an object P of $\mathcal{C}$ together with a family of morphisms $\{\pi_i : P \to A_i \mid i \in I\}$ such that for any object B and family of morphisms $\{\varphi_i : B \to A_i \mid i \in I\}$ in $\mathcal{C}$, there is a unique morphism $\varphi : B \to P$

Products in a Category

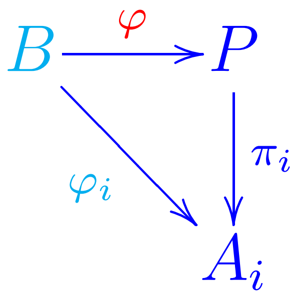
Definition (7.2). Let $\mathcal{C}$ be a category and let $\{A_i \mid i \in I\}$ be a family of objects of $\mathcal{C}$. A product for the family $\{A_i \mid i \in I\}$ is an object P of $\mathcal{C}$ together with a family of morphisms $\{\pi_i : P \longrightarrow A_i \mid i \in I\}$ such that for any object B and family of morphisms $\{\varphi_i : B \longrightarrow A_i \mid i \in I\}$ in $\mathcal{C}$, there is a unique morphism $\varphi : B \longrightarrow P$ such that the diagram



commutes for all $i \in I$.

Products in a Category

Definition (7.2). Let $\mathcal{C}$ be a category and let $\{A_i \mid i \in I\}$ be a family of objects of $\mathcal{C}$. A product for the family $\{A_i \mid i \in I\}$ is an object P of $\mathcal{C}$ together with a family of morphisms $\{\pi_i : P \to A_i \mid i \in I\}$ such that for any object B and family of morphisms $\{\varphi_i : B \to A_i \mid i \in I\}$ in $\mathcal{C}$, there is a unique morphism $\varphi : B \to P$ such that the diagram

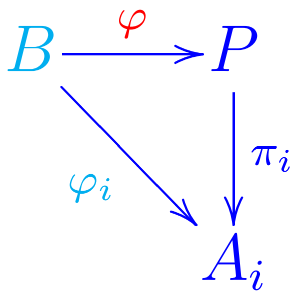


commutes for all $i \in I$.

Example. Let $\{A_i \mid i \in I\}$ be a family of sets.

Products in a Category

Definition (7.2). Let $\mathcal{C}$ be a category and let $\{A_i \mid i \in I\}$ be a family of objects of $\mathcal{C}$. A product for the family $\{A_i \mid i \in I\}$ is an object P of $\mathcal{C}$ together with a family of morphisms $\{\pi_i : P \to A_i \mid i \in I\}$ such that for any object B and family of morphisms $\{\varphi_i : B \to A_i \mid i \in I\}$ in $\mathcal{C}$, there is a unique morphism $\varphi : B \to P$ such that the diagram

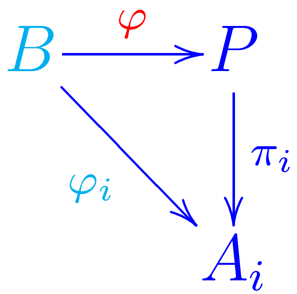


commutes for all $i \in I$.

Example. Let $\{A_i \mid i \in I\}$ be a family of sets. The Cartesian product $\prod_{i \in I} A_i$

Products in a Category

Definition (7.2). Let $\mathcal{C}$ be a category and let $\{A_i \mid i \in I\}$ be a family of objects of $\mathcal{C}$. A product for the family $\{A_i \mid i \in I\}$ is an object P of $\mathcal{C}$ together with a family of morphisms $\{\pi_i : P \to A_i \mid i \in I\}$ such that for any object B and family of morphisms $\{\varphi_i : B \to A_i \mid i \in I\}$ in $\mathcal{C}$, there is a unique morphism $\varphi : B \to P$ such that the diagram

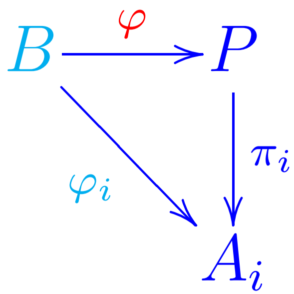


commutes for all $i \in I$.

Example. Let $\{A_i \mid i \in I\}$ be a family of sets. The Cartesian product $\prod_{i \in I} A_i$ together with the canonical projections $\{\pi_i : \prod_{i \in I} A_i \to A_i \mid i \in I\}$

Products in a Category

Definition (7.2)**.** Let $\mathcal{C}$ be a category and let $\{A_i \mid i \in I\}$ be a family of objects of $\mathcal{C}$. A product for the family $\{A_i \mid i \in I\}$ is an object P of $\mathcal{C}$ together with a family of morphisms $\{\pi_i : P \to A_i \mid i \in I\}$ such that for any object B and family of morphisms $\{\varphi_i : B \to A_i \mid i \in I\}$ in $\mathcal{C}$, there is a unique morphism $\varphi : B \to P$ such that the diagram



commutes for all $i \in I$.

Example. Let $\{A_i \mid i \in I\}$ be a family of sets. The Cartesian product $\prod_{i\in I} A_i$ together with the canonical projections $\{\pi_i : \prod_{i\in I} A_i \to A_i \mid i \in I\}$ is a product for the family $\{A_i \mid i \in I\}$

Products in a Category

Definition (7.2). Let $\mathcal{C}$ be a category and let $\{A_i \mid i \in I\}$ be a family of objects of $\mathcal{C}$. A product for the family $\{A_i \mid i \in I\}$ is an object P of $\mathcal{C}$ together with a family of morphisms $\{\pi_i : P \to A_i \mid i \in I\}$ such that for any object B and family of morphisms $\{\varphi_i : B \to A_i \mid i \in I\}$ in $\mathcal{C}$, there is a unique morphism $\varphi : B \to P$ such that the diagram

$$\begin{array}{ccc} B & \xrightarrow{\varphi} & P \\ & {\scriptstyle\varphi_i}\searrow & \downarrow{\scriptstyle\pi_i} \\ & & A_i \end{array}$$

commutes for all $i \in I$.

Example. Let $\{A_i \mid i \in I\}$ be a family of sets. The Cartesian product $\prod_{i \in I} A_i$ together with the canonical projections $\{\pi_i : \prod_{i \in I} A_i \to A_i \mid i \in I\}$ is a product for the family $\{A_i \mid i \in I\}$ in the category $\mathcal{S}$ of sets.

Theorem (7.3)

Let $\{A_i \mid i \in I\}$ be a family of objects of a category $\mathcal{C}$.

Theorem (7.3)

Let $\{A_i \mid i \in I\}$ be a family of objects of a category $\mathcal{C}$. Suppose $(P, \{\pi_i\})$ and $(Q, \{\psi_i\})$ are both products of the family $\{A_i \mid i \in I\}$.

Theorem (7.3)

Let $\{A_i \mid i \in I\}$ be a family of objects of a category $\mathcal{C}$. Suppose $(P, \{\pi_i\})$ and $(Q, \{\psi_i\})$ are both products of the family $\{A_i \mid i \in I\}$. Then P and Q are equivalent in $\mathcal{C}$.

Theorem (7.3)

Let $\{A_i \mid i \in I\}$ be a family of objects of a category $\mathcal{C}$. Suppose $(P, \{\pi_i\})$ and $(Q, \{\psi_i\})$ are both products of the family $\{A_i \mid i \in I\}$. Then P and Q are equivalent in $\mathcal{C}$.

Proof.

Theorem (7.3)

Let $\{A_i \mid i \in I\}$ be a family of objects of a category $\mathcal{C}$. Suppose $(P, \{\pi_i\})$ and $(Q, \{\psi_i\})$ are both products of the family $\{A_i \mid i \in I\}$. Then P and Q are equivalent in $\mathcal{C}$.

Proof.

Since $(P, \{\pi_i\})$ is a product of the family $\{A_i \mid i \in I\}$,

Theorem (7.3)

Let $\{A_i \mid i \in I\}$ be a family of objects of a category $\mathcal{C}$. Suppose $(P, \{\pi_i\})$ and $(Q, \{\psi_i\})$ are both products of the family $\{A_i \mid i \in I\}$. Then P and Q are equivalent in $\mathcal{C}$.

Proof.

Since $(P, \{\pi_i\})$ is a product of the family $\{A_i \mid i \in I\}$, for the object Q

Theorem (7.3)

Let $\{A_i \mid i \in I\}$ be a family of objects of a category $\mathcal{C}$. Suppose $(P, \{\pi_i\})$ and $(Q, \{\psi_i\})$ are both products of the family $\{A_i \mid i \in I\}$. Then P and Q are equivalent in $\mathcal{C}$.

Proof.

Since $(P, \{\pi_i\})$ is a product of the family $\{A_i \mid i \in I\}$, for the object Q and the family of morphisms $\{\psi_i : Q \to A_i \mid i \in I\}$,

Theorem (7.3)

Let $\{A_i \mid i \in I\}$ be a family of objects of a category $\mathcal{C}$. Suppose $(P, \{\pi_i\})$ and $(Q, \{\psi_i\})$ are both products of the family $\{A_i \mid i \in I\}$. Then P and Q are equivalent in $\mathcal{C}$.

Proof.

Since $(P, \{\pi_i\})$ is a product of the family $\{A_i \mid i \in I\}$, for the object Q and the family of morphisms $\{\psi_i : Q \to A_i \mid i \in I\}$, there exists a morphism $\psi : Q \to P$

Theorem (7.3)

Let $\{A_i \mid i \in I\}$ be a family of objects of a category $\mathcal{C}$. Suppose $(P, \{\pi_i\})$ and $(Q, \{\psi_i\})$ are both products of the family $\{A_i \mid i \in I\}$. Then P and Q are equivalent in $\mathcal{C}$.

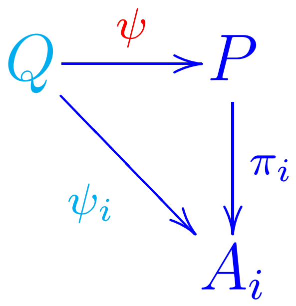
Proof.

Since $(P, \{\pi_i\})$ is a product of the family $\{A_i \mid i \in I\}$, for the object Q and the family of morphisms $\{\psi_i : Q \to A_i \mid i \in I\}$, there exists a morphism $\psi : Q \to P$ such that the above diagram

Theorem (7.3)

Let $\{A_i \mid i \in I\}$ be a family of objects of a category $\mathcal{C}$. Suppose $(P, \{\pi_i\})$ and $(Q, \{\psi_i\})$ are both products of the family $\{A_i \mid i \in I\}$. Then P and Q are equivalent in $\mathcal{C}$.

Proof.

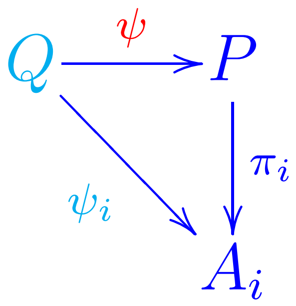


Since $(P, \{\pi_i\})$ is a product of the family $\{A_i \mid i \in I\}$, for the object Q and the family of morphisms $\{\psi_i : Q \to A_i \mid i \in I\}$, there exists a morphism $\psi : Q \to P$ such that the above diagram

Theorem (7.3)

Let $\{A_i \mid i \in I\}$ be a family of objects of a category $\mathcal{C}$. Suppose $(P, \{\pi_i\})$ and $(Q, \{\psi_i\})$ are both products of the family $\{A_i \mid i \in I\}$. Then P and Q are equivalent in $\mathcal{C}$.

Proof.

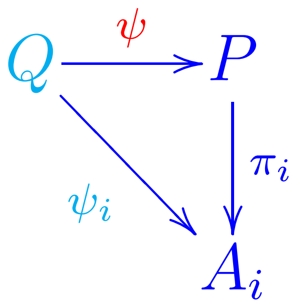


Since $(P, \{\pi_i\})$ is a product of the family $\{A_i \mid i \in I\}$, for the object Q and the family of morphisms $\{\psi_i : Q \to A_i \mid i \in I\}$, there exists a morphism $\psi : Q \to P$ such that the above diagram commutes for all $i \in I$.

Theorem (7.3)

Let $\{A_i \mid i \in I\}$ be a family of objects of a category $\mathcal{C}$. Suppose $(P, \{\pi_i\})$ and $(Q, \{\psi_i\})$ are both products of the family $\{A_i \mid i \in I\}$. Then P and Q are equivalent in $\mathcal{C}$.

Proof.

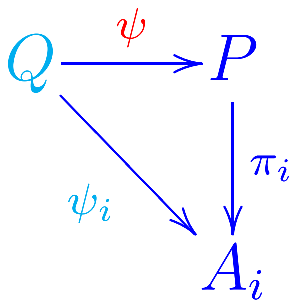


Since $(P, \{\pi_i\})$ is a product of the family $\{A_i \mid i \in I\}$, for the object Q and the family of morphisms $\{\psi_i : Q \to A_i \mid i \in I\}$, there exists a morphism $\psi : Q \to P$ such that the above diagram commutes for all $i \in I$. Similarly, since $(Q, \{\psi_i\})$ is also a product of $\{A_i \mid i \in I\}$,

Theorem (7.3)

Let $\{A_i \mid i \in I\}$ be a family of objects of a category $\mathcal{C}$. Suppose $(P, \{\pi_i\})$ and $(Q, \{\psi_i\})$ are both products of the family $\{A_i \mid i \in I\}$. Then P and Q are equivalent in $\mathcal{C}$.

Proof.

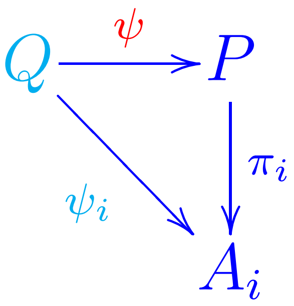


Since $(P, \{\pi_i\})$ is a product of the family $\{A_i \mid i \in I\}$, for the object Q and the family of morphisms $\{\psi_i : Q \to A_i \mid i \in I\}$, there exists a morphism $\psi : Q \to P$ such that the above diagram commutes for all $i \in I$. Similarly, since $(Q, \{\psi_i\})$ is also a product of $\{A_i \mid i \in I\}$, there exists a morphism $\pi : P \to Q$

Theorem (7.3)

Let $\{A_i \mid i \in I\}$ be a family of objects of a category $\mathcal{C}$. Suppose $(P, \{\pi_i\})$ and $(Q, \{\psi_i\})$ are both products of the family $\{A_i \mid i \in I\}$. Then P and Q are equivalent in $\mathcal{C}$.

Proof.

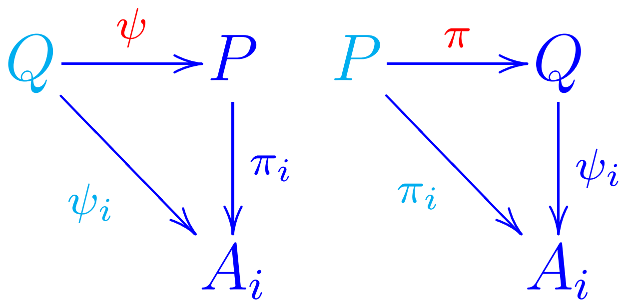


Since $(P, \{\pi_i\})$ is a product of the family $\{A_i \mid i \in I\}$, for the object Q and the family of morphisms $\{\psi_i : Q \to A_i \mid i \in I\}$, there exists a morphism $\psi : Q \to P$ such that the above diagram commutes for all $i \in I$. Similarly, since $(Q, \{\psi_i\})$ is also a product of $\{A_i \mid i \in I\}$, there exists a morphism $\pi : P \to Q$ such that the second diagram above

Theorem (7.3)

Let $\{A_i \mid i \in I\}$ be a family of objects of a category $\mathcal{C}$. Suppose $(P, \{\pi_i\})$ and $(Q, \{\psi_i\})$ are both products of the family $\{A_i \mid i \in I\}$. Then P and Q are equivalent in $\mathcal{C}$.

Proof.

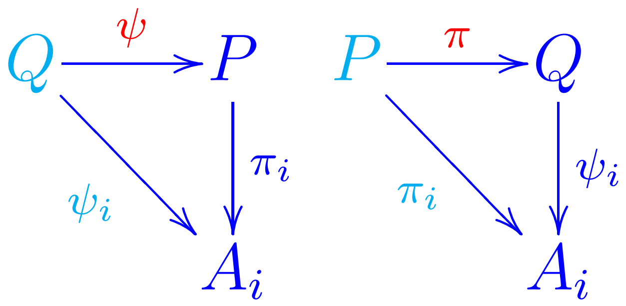


Since $(P, \{\pi_i\})$ is a product of the family $\{A_i \mid i \in I\}$, for the object Q and the family of morphisms $\{\psi_i : Q \to A_i \mid i \in I\}$, there exists a morphism $\psi : Q \to P$ such that the above diagram commutes for all $i \in I$. Similarly, since $(Q, \{\psi_i\})$ is also a product of $\{A_i \mid i \in I\}$, there exists a morphism $\pi : P \to Q$ such that the second diagram above

Theorem (7.3)

Let $\{A_i \mid i \in I\}$ be a family of objects of a category $\mathcal{C}$. Suppose $(P, \{\pi_i\})$ and $(Q, \{\psi_i\})$ are both products of the family $\{A_i \mid i \in I\}$. Then P and Q are equivalent in $\mathcal{C}$.

Proof.

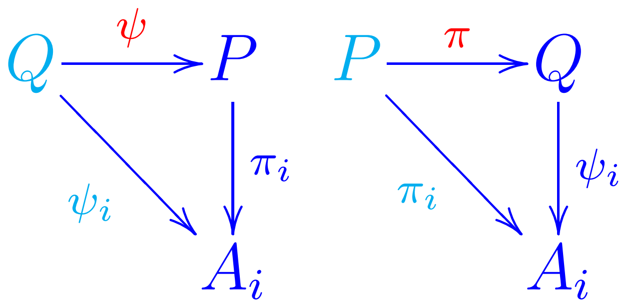


Since $(P, \{\pi_i\})$ is a product of the family $\{A_i \mid i \in I\}$, for the object Q and the family of morphisms $\{\psi_i : Q \to A_i \mid i \in I\}$, there exists a morphism $\psi : Q \to P$ such that the above diagram commutes for all $i \in I$. Similarly, since $(Q, \{\psi_i\})$ is also a product of $\{A_i \mid i \in I\}$, there exists a morphism $\pi : P \to Q$ such that the second diagram above commutes for all $i \in I$.

Theorem (7.3)

Let $\{A_i \mid i \in I\}$ be a family of objects of a category $\mathcal{C}$. Suppose $(P, \{\pi_i\})$ and $(Q, \{\psi_i\})$ are both products of the family $\{A_i \mid i \in I\}$. Then P and Q are equivalent in $\mathcal{C}$.

Proof.

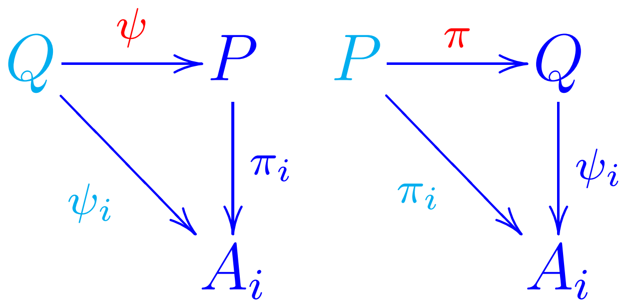


Combine the above two commutative diagrams

Theorem (7.3)

Let $\{A_i \mid i \in I\}$ be a family of objects of a category $\mathcal{C}$. Suppose $(P, \{\pi_i\})$ and $(Q, \{\psi_i\})$ are both products of the family $\{A_i \mid i \in I\}$. Then P and Q are equivalent in $\mathcal{C}$.

Proof.

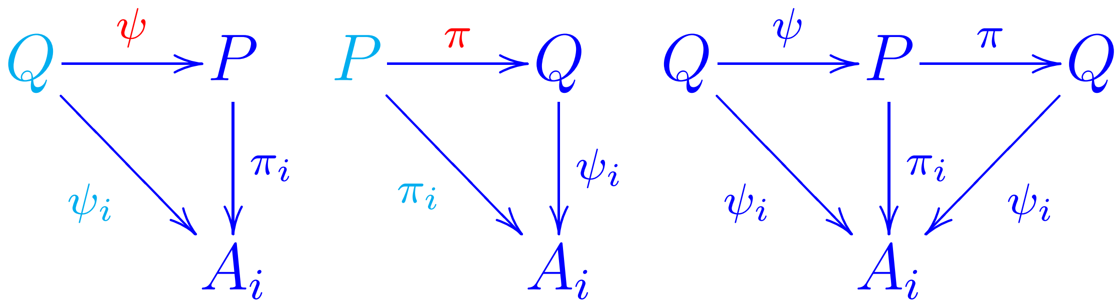


Combine the above two commutative diagrams and we have the third commutative diagram above

Theorem (7.3)

Let $\{A_i \mid i \in I\}$ be a family of objects of a category $\mathcal{C}$. Suppose $(P, \{\pi_i\})$ and $(Q, \{\psi_i\})$ are both products of the family $\{A_i \mid i \in I\}$. Then P and Q are equivalent in $\mathcal{C}$.

Proof.

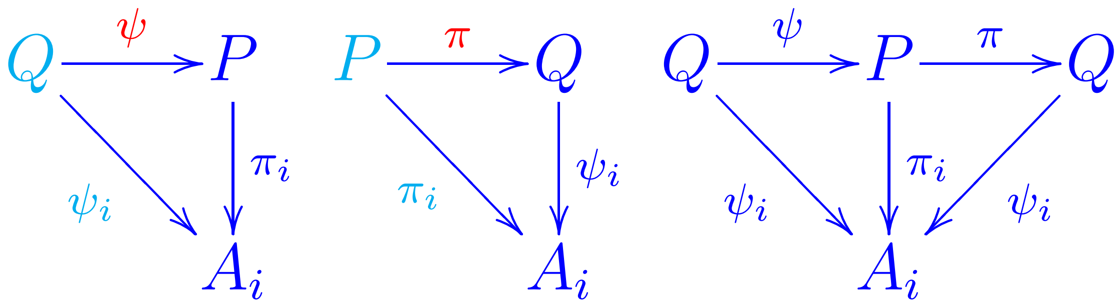


Combine the above two commutative diagrams and we have the third commutative diagram above

Theorem (7.3)

Let $\{A_i \mid i \in I\}$ be a family of objects of a category $\mathcal{C}$. Suppose $(P, \{\pi_i\})$ and $(Q, \{\psi_i\})$ are both products of the family $\{A_i \mid i \in I\}$. Then P and Q are equivalent in $\mathcal{C}$.

Proof.

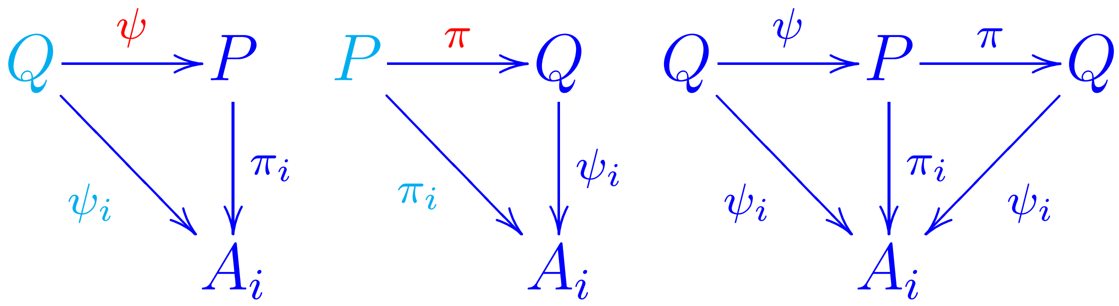


Combine the above two commutative diagrams and we have the third commutative diagram above for all $i \in I$.

Theorem (7.3)

Let $\{A_i \mid i \in I\}$ be a family of objects of a category $\mathcal{C}$. Suppose $(P, \{\pi_i\})$ and $(Q, \{\psi_i\})$ are both products of the family $\{A_i \mid i \in I\}$. Then P and Q are equivalent in $\mathcal{C}$.

Proof.

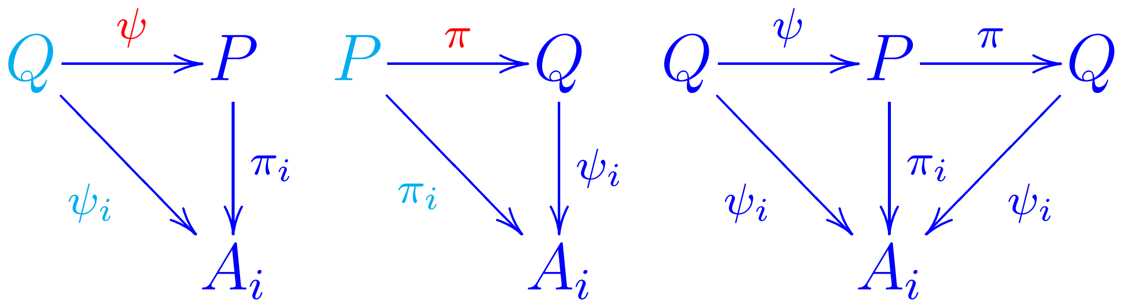


Combine the above two commutative diagrams and we have the third commutative diagram above for all $i \in I$. In other words,

Theorem (7.3)

Let $\{A_i \mid i \in I\}$ be a family of objects of a category $\mathcal{C}$. Suppose $(P, \{\pi_i\})$ and $(Q, \{\psi_i\})$ are both products of the family $\{A_i \mid i \in I\}$. Then P and Q are equivalent in $\mathcal{C}$.

Proof.

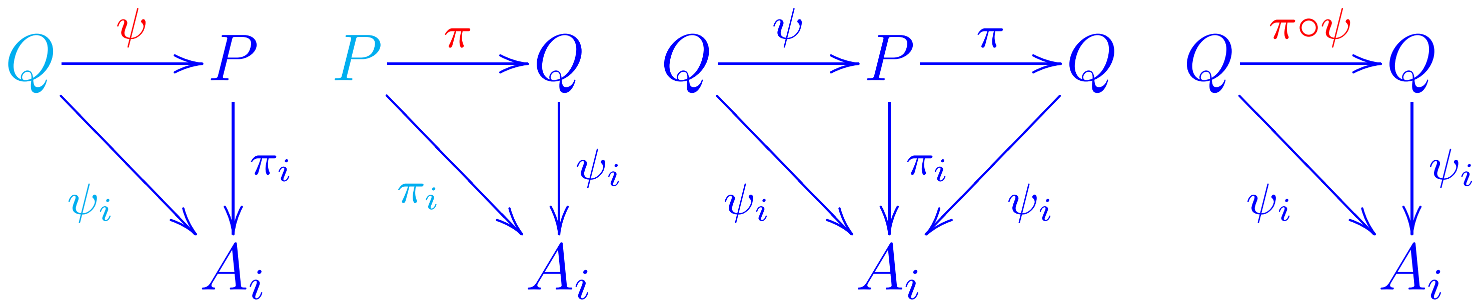


Combine the above two commutative diagrams and we have the third commutative diagram above for all $i \in I$. In other words, we have the fourth commutative diagram above

Theorem (7.3)

Let $\{A_i \mid i \in I\}$ be a family of objects of a category $\mathcal{C}$. Suppose $(P, \{\pi_i\})$ and $(Q, \{\psi_i\})$ are both products of the family $\{A_i \mid i \in I\}$. Then P and Q are equivalent in $\mathcal{C}$.

Proof.

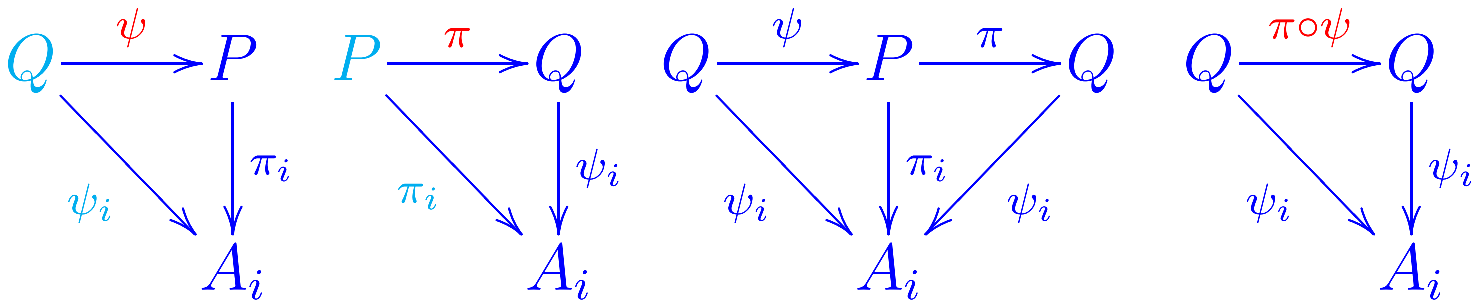


Combine the above two commutative diagrams and we have the third commutative diagram above for all $i \in I$. In other words, we have the fourth commutative diagram above

Theorem (7.3)

Let $\{A_i \mid i \in I\}$ be a family of objects of a category $\mathcal{C}$. Suppose $(P, \{\pi_i\})$ and $(Q, \{\psi_i\})$ are both products of the family $\{A_i \mid i \in I\}$. Then P and Q are equivalent in $\mathcal{C}$.

Proof.

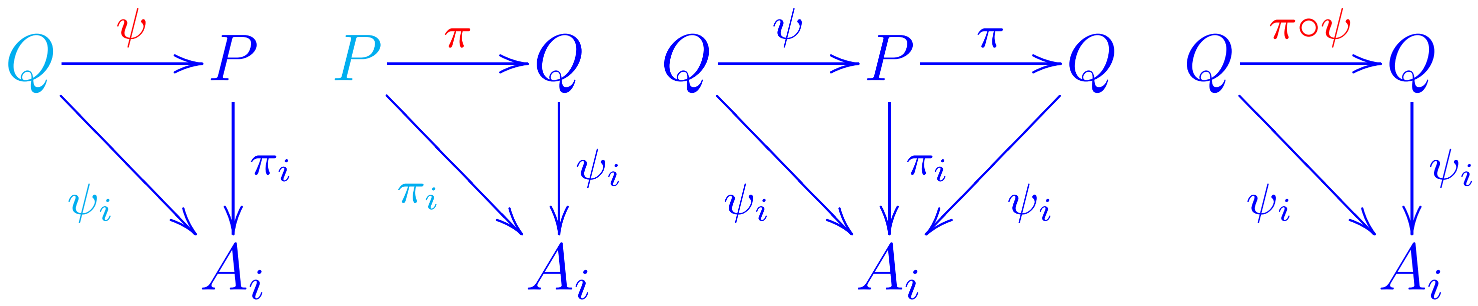


Combine the above two commutative diagrams and we have the third commutative diagram above for all $i \in I$. In other words, we have the fourth commutative diagram above for all $i \in I$.

Theorem (7.3)

Let $\{A_i \mid i \in I\}$ be a family of objects of a category $\mathcal{C}$. Suppose $(P, \{\pi_i\})$ and $(Q, \{\psi_i\})$ are both products of the family $\{A_i \mid i \in I\}$. Then P and Q are equivalent in $\mathcal{C}$.

Proof.

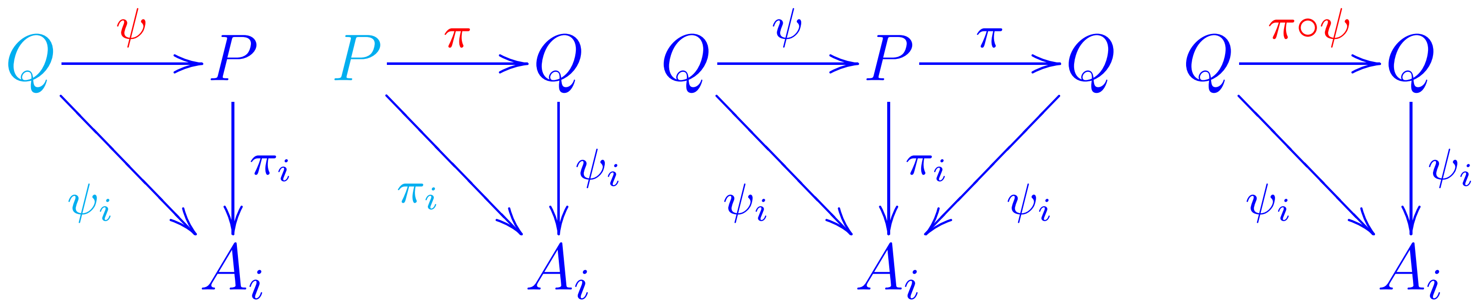


Combine the above two commutative diagrams and we have the third commutative diagram above for all $i \in I$. In other words, we have the fourth commutative diagram above for all $i \in I$. On the other hand,

Theorem (7.3)

Let $\{A_i \mid i \in I\}$ be a family of objects of a category $\mathcal{C}$. Suppose $(P, \{\pi_i\})$ and $(Q, \{\psi_i\})$ are both products of the family $\{A_i \mid i \in I\}$. Then P and Q are equivalent in $\mathcal{C}$.

Proof.

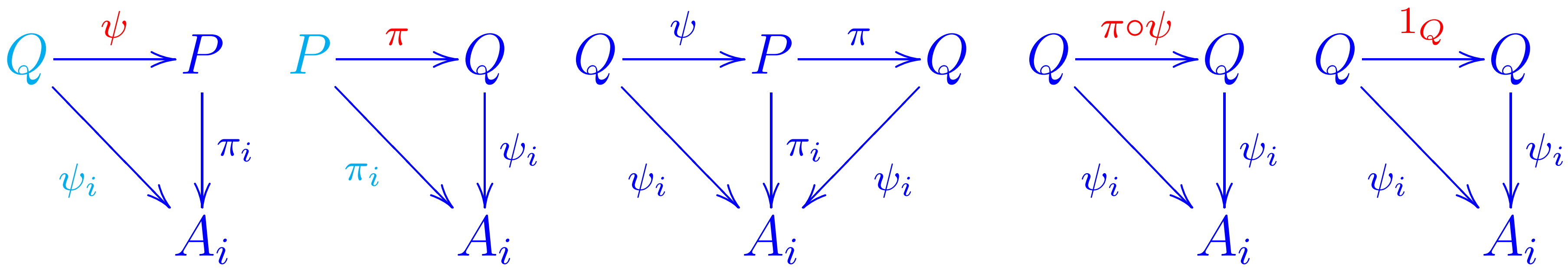


Combine the above two commutative diagrams and we have the third commutative diagram above for all $i \in I$. In other words, we have the fourth commutative diagram above for all $i \in I$. On the other hand, it is clear that we have the fifth commutative diagram above.

Theorem (7.3)

Let $\{A_i \mid i \in I\}$ be a family of objects of a category $\mathcal{C}$. Suppose $(P, \{\pi_i\})$ and $(Q, \{\psi_i\})$ are both products of the family $\{A_i \mid i \in I\}$. Then P and Q are equivalent in $\mathcal{C}$.

Proof.

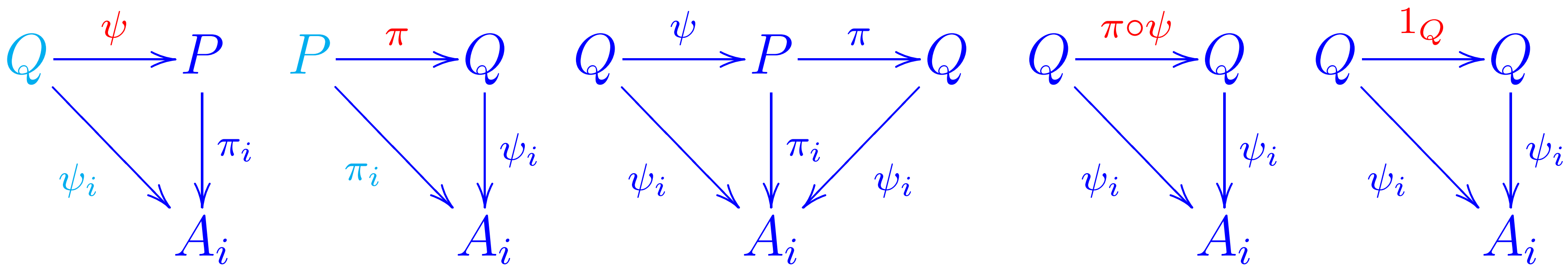


Combine the above two commutative diagrams and we have the third commutative diagram above for all $i \in I$. In other words, we have the fourth commutative diagram above for all $i \in I$. On the other hand, it is clear that we have the fifth commutative diagram above.

Theorem (7.3)

Let $\{A_i \mid i \in I\}$ be a family of objects of a category $\mathcal{C}$. Suppose $(P, \{\pi_i\})$ and $(Q, \{\psi_i\})$ are both products of the family $\{A_i \mid i \in I\}$. Then P and Q are equivalent in $\mathcal{C}$.

Proof.

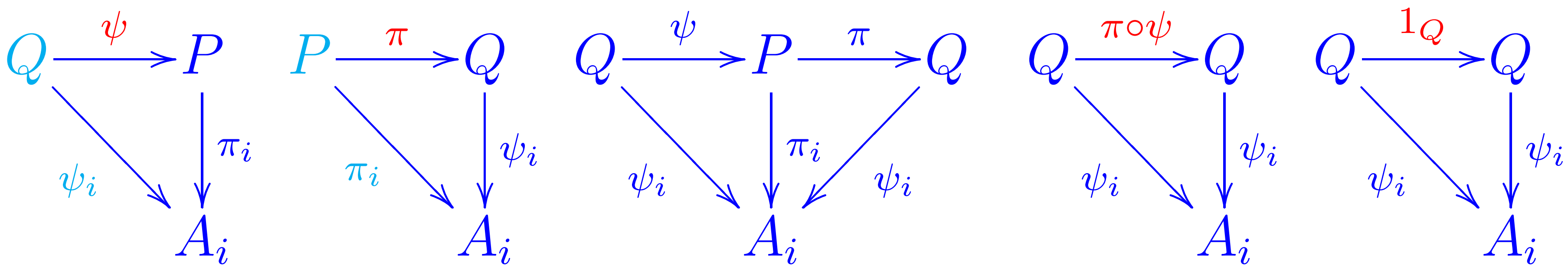


Combine the above two commutative diagrams and we have the third commutative diagram above for all $i \in I$. In other words, we have the fourth commutative diagram above for all $i \in I$. On the other hand, it is clear that we have the fifth commutative diagram above. By the uniqueness of the morphism $Q \to Q$ that makes the diagram commute,

Theorem (7.3)

Let $\{A_i \mid i \in I\}$ be a family of objects of a category $\mathcal{C}$. Suppose $(P, \{\pi_i\})$ and $(Q, \{\psi_i\})$ are both products of the family $\{A_i \mid i \in I\}$. Then P and Q are equivalent in $\mathcal{C}$.

Proof.

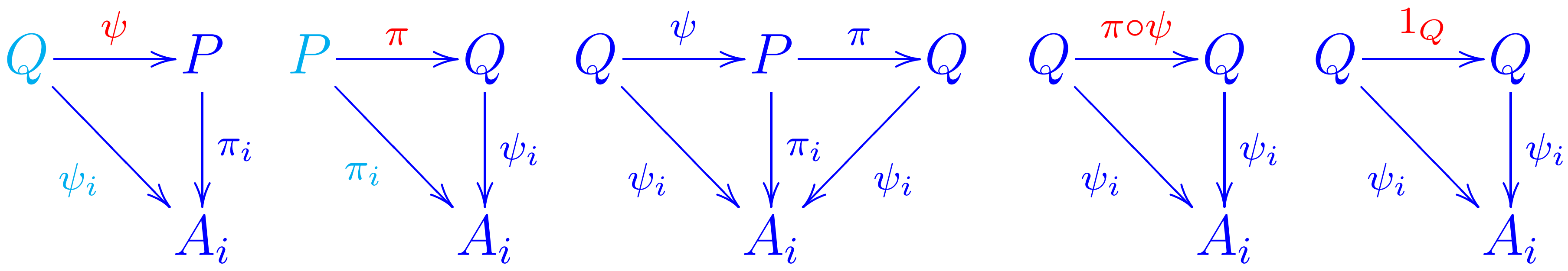


Combine the above two commutative diagrams and we have the third commutative diagram above for all $i \in I$. In other words, we have the fourth commutative diagram above for all $i \in I$. On the other hand, it is clear that we have the fifth commutative diagram above. By the uniqueness of the morphism $Q \to Q$ that makes the diagram commute, we have $\pi \circ \psi = 1_Q$.

Theorem (7.3)

Let $\{A_i \mid i \in I\}$ be a family of objects of a category $\mathcal{C}$. Suppose $(P, \{\pi_i\})$ and $(Q, \{\psi_i\})$ are both products of the family $\{A_i \mid i \in I\}$. Then P and Q are equivalent in $\mathcal{C}$.

Proof. We have $\pi \circ \psi = 1_Q$

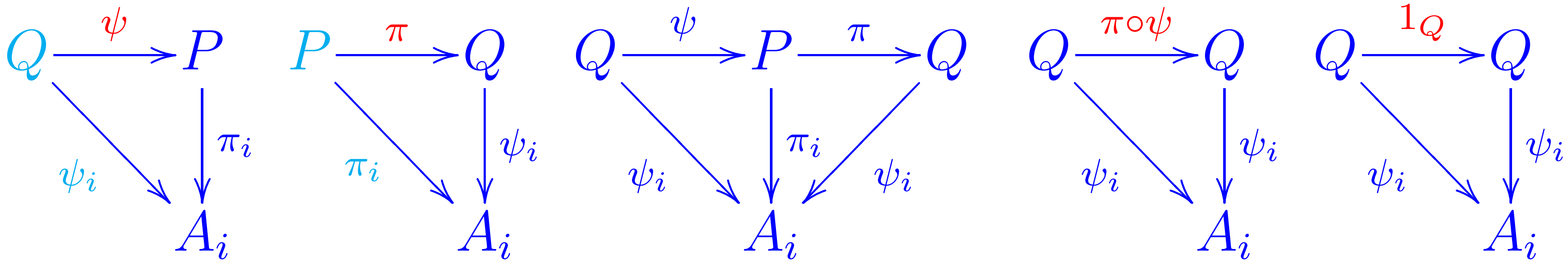


Combine the above two commutative diagrams and we have the third commutative diagram above for all $i \in I$. In other words, we have the fourth commutative diagram above for all $i \in I$. On the other hand, it is clear that we have the fifth commutative diagram above. By the uniqueness of the morphism $Q \to Q$ that makes the diagram commute, we have $\pi \circ \psi = 1_Q$.

Theorem (7.3)

Let $\{A_i \mid i \in I\}$ be a family of objects of a category $\mathcal{C}$. Suppose $(P, \{\pi_i\})$ and $(Q, \{\psi_i\})$ are both products of the family $\{A_i \mid i \in I\}$. Then P and Q are equivalent in $\mathcal{C}$.

Proof. We have $\pi \circ \psi = 1_Q$

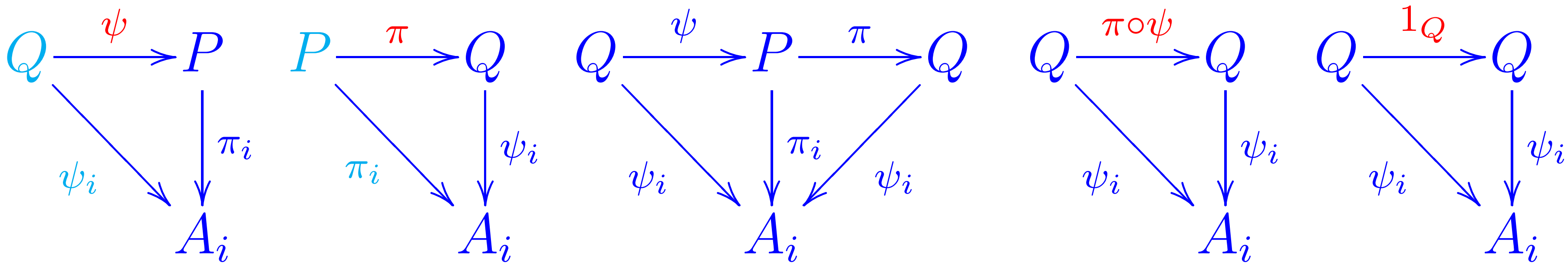


With a similar discussion,

Theorem (7.3)

Let $\{A_i \mid i \in I\}$ be a family of objects of a category $\mathcal{C}$. Suppose $(P, \{\pi_i\})$ and $(Q, \{\psi_i\})$ are both products of the family $\{A_i \mid i \in I\}$. Then P and Q are equivalent in $\mathcal{C}$.

Proof. We have $\pi \circ \psi = 1_Q$

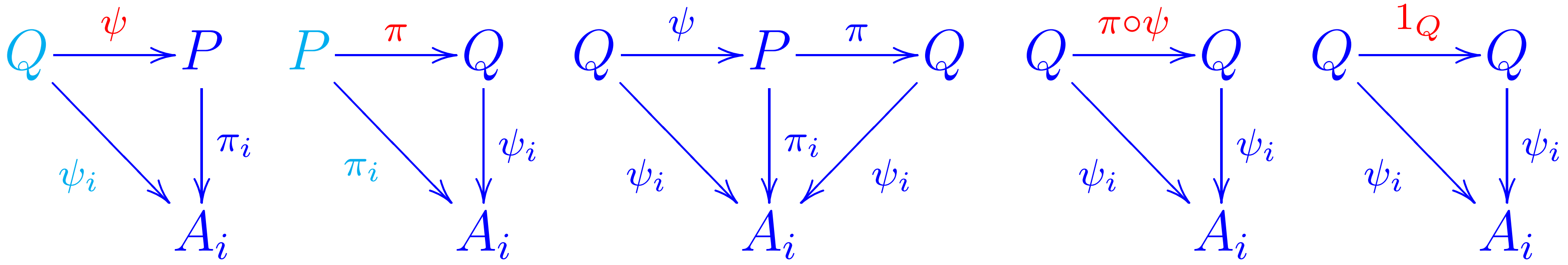


With a similar discussion, we will get $\psi \circ \pi = 1_P$.

Theorem (7.3)

Let $\{A_i \mid i \in I\}$ be a family of objects of a category $\mathcal{C}$. Suppose $(P, \{\pi_i\})$ and $(Q, \{\psi_i\})$ are both products of the family $\{A_i \mid i \in I\}$. Then P and Q are equivalent in $\mathcal{C}$.

Proof. We have $\pi \circ \psi = 1_Q$

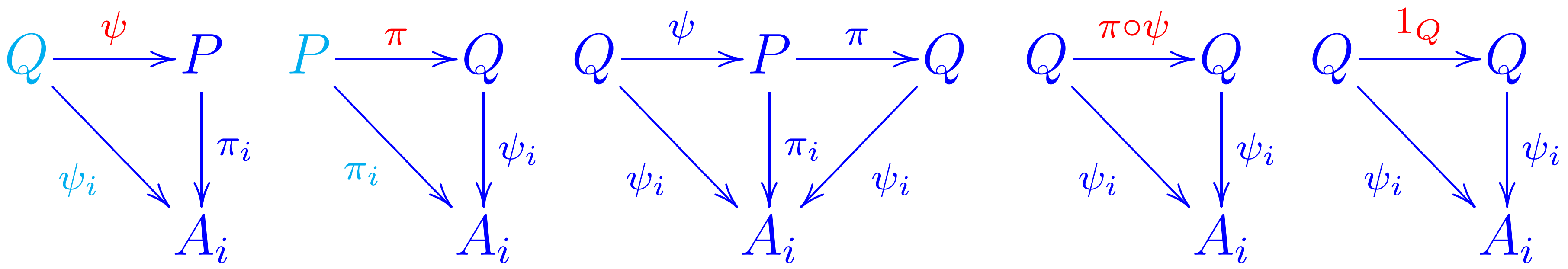


More precisely,

Theorem (7.3)

Let $\{A_i \mid i \in I\}$ be a family of objects of a category $\mathcal{C}$. Suppose $(P, \{\pi_i\})$ and $(Q, \{\psi_i\})$ are both products of the family $\{A_i \mid i \in I\}$. Then P and Q are equivalent in $\mathcal{C}$.

Proof. We have $\pi \circ \psi = 1_Q$

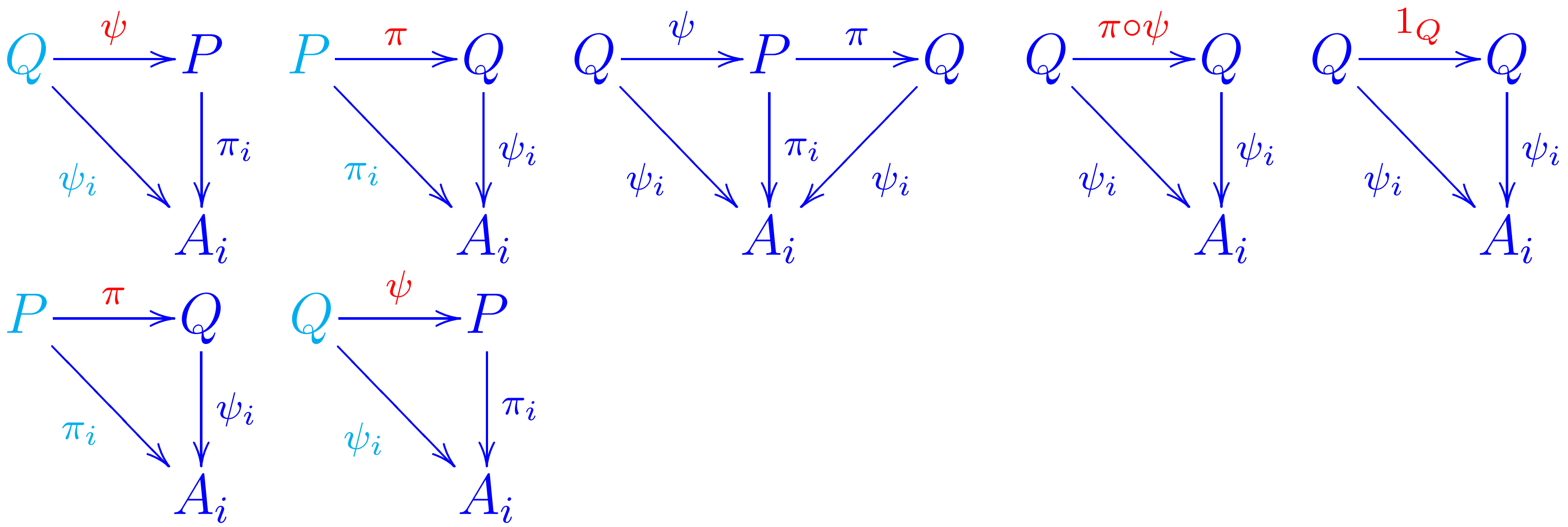


More precisely, combine these commutative diagrams,

Theorem (7.3)

Let $\{A_i \mid i \in I\}$ be a family of objects of a category $\mathcal{C}$. Suppose $(P, \{\pi_i\})$ and $(Q, \{\psi_i\})$ are both products of the family $\{A_i \mid i \in I\}$. Then P and Q are equivalent in $\mathcal{C}$.

Proof. We have $\pi \circ \psi = 1_Q$

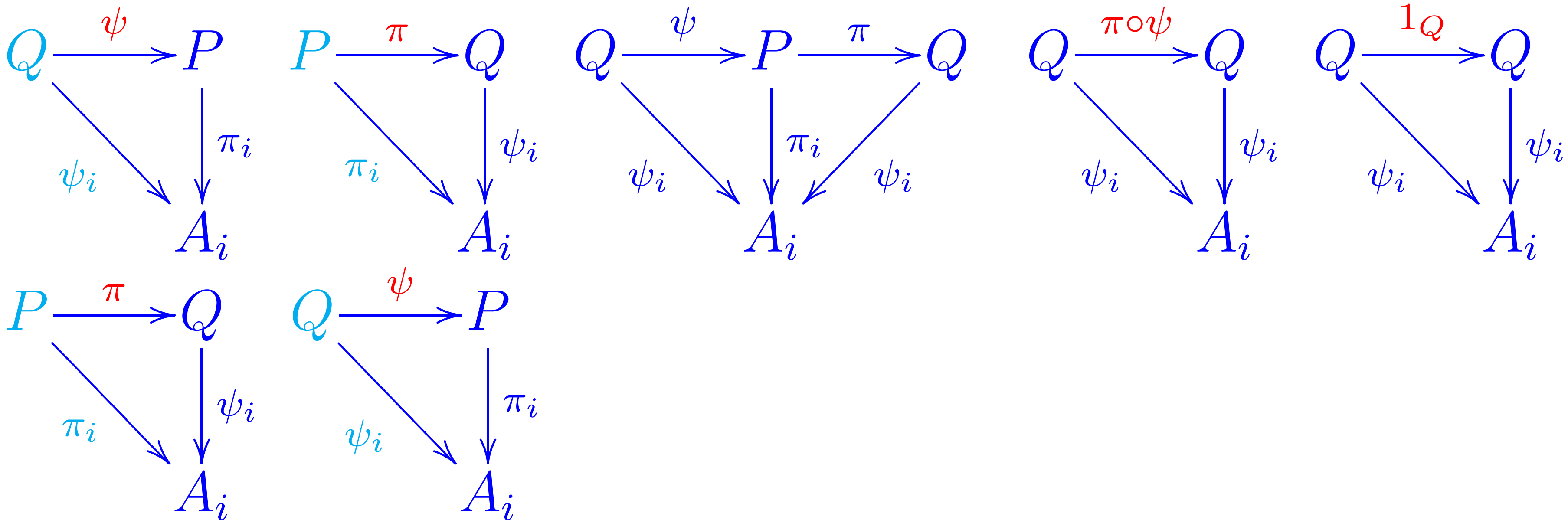


More precisely, combine these commutative diagrams,

Theorem (7.3)

Let $\{A_i \mid i \in I\}$ be a family of objects of a category $\mathcal{C}$. Suppose $(P, \{\pi_i\})$ and $(Q, \{\psi_i\})$ are both products of the family $\{A_i \mid i \in I\}$. Then P and Q are equivalent in $\mathcal{C}$.

Proof. We have $\pi \circ \psi = 1_Q$

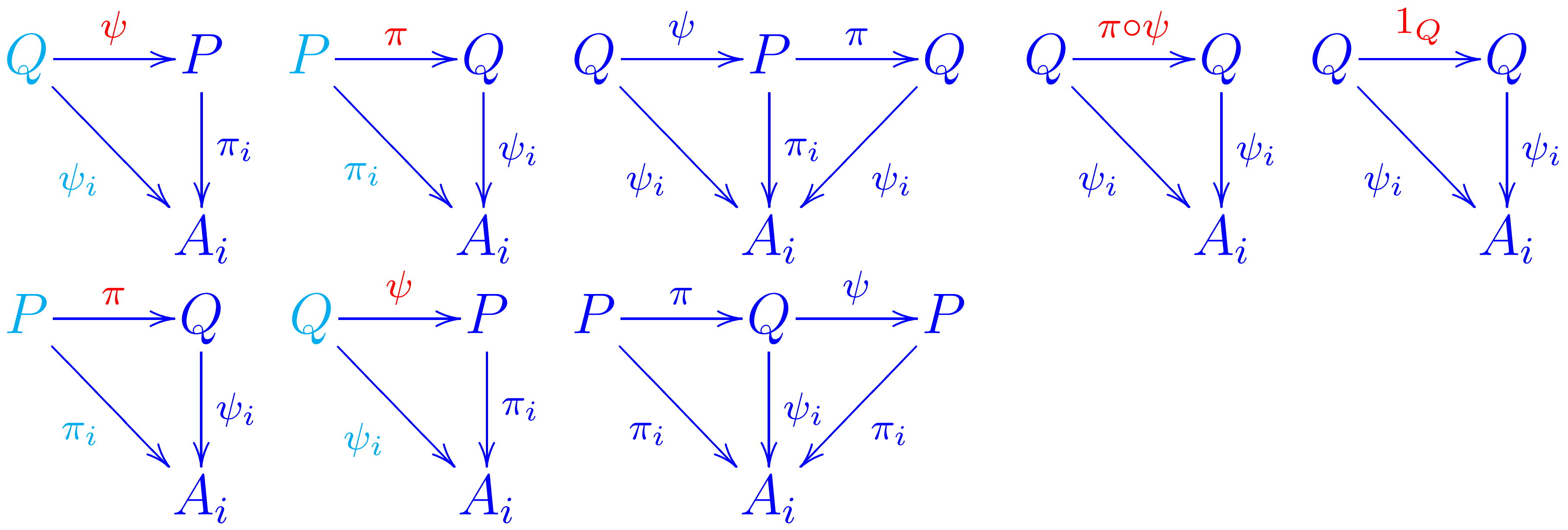


More precisely, combine these commutative diagrams, then we get this commutative diagram,

Theorem (7.3)

Let $\{A_i \mid i \in I\}$ be a family of objects of a category $\mathcal{C}$. Suppose $(P, \{\pi_i\})$ and $(Q, \{\psi_i\})$ are both products of the family $\{A_i \mid i \in I\}$. Then P and Q are equivalent in $\mathcal{C}$.

Proof. We have $\pi \circ \psi = 1_Q$

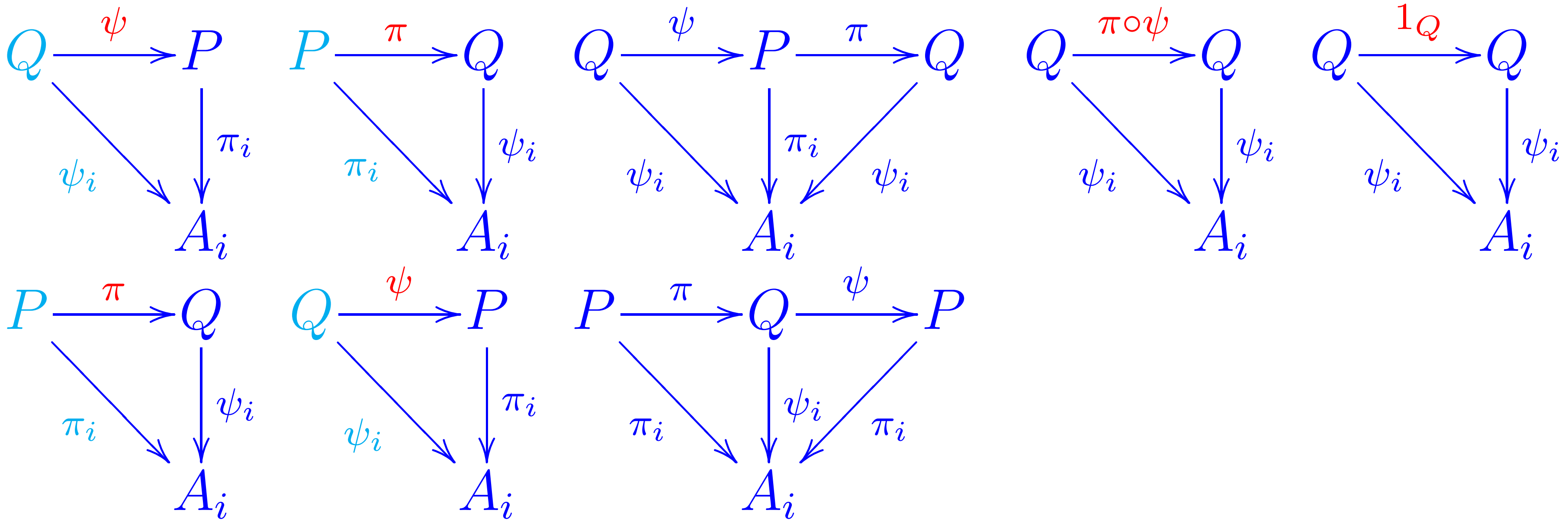


More precisely, combine these commutative diagrams, then we get this commutative diagram,

Theorem (7.3)

Let $\{A_i \mid i \in I\}$ be a family of objects of a category $\mathcal{C}$. Suppose $(P, \{\pi_i\})$ and $(Q, \{\psi_i\})$ are both products of the family $\{A_i \mid i \in I\}$. Then P and Q are equivalent in $\mathcal{C}$.

Proof. We have $\pi \circ \psi = 1_Q$

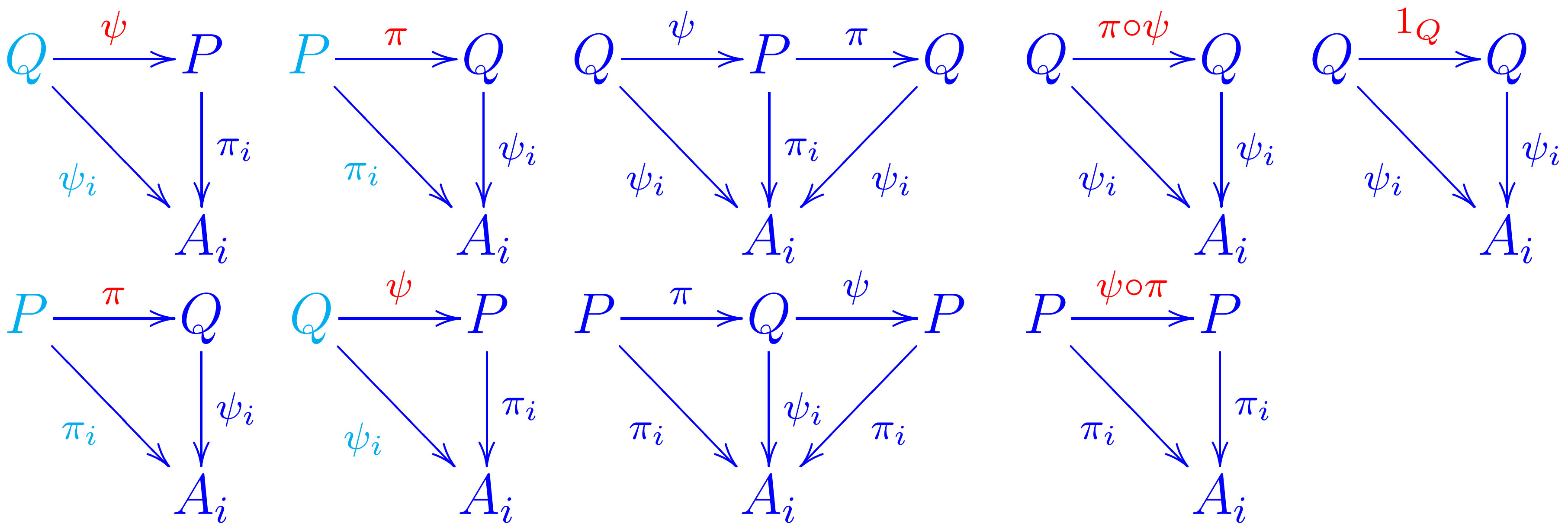


More precisely, combine these commutative diagrams, then we get this commutative diagram, and so this commutative diagram.

Theorem (7.3)

Let $\{A_i \mid i \in I\}$ be a family of objects of a category $\mathcal{C}$. Suppose $(P, \{\pi_i\})$ and $(Q, \{\psi_i\})$ are both products of the family $\{A_i \mid i \in I\}$. Then P and Q are equivalent in $\mathcal{C}$.

Proof. We have $\pi \circ \psi = 1_Q$

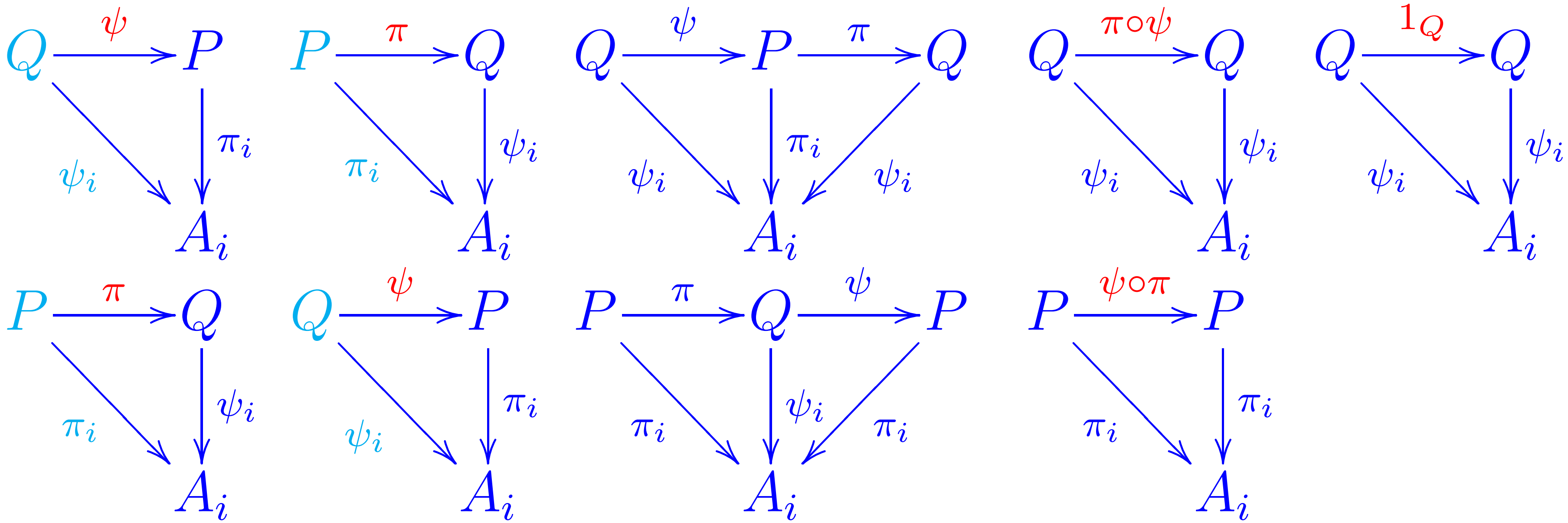


More precisely, combine these commutative diagrams, then we get this commutative diagram, and so this commutative diagram.

Theorem (7.3)

Let $\{A_i \mid i \in I\}$ be a family of objects of a category $\mathcal{C}$. Suppose $(P, \{\pi_i\})$ and $(Q, \{\psi_i\})$ are both products of the family $\{A_i \mid i \in I\}$. Then P and Q are equivalent in $\mathcal{C}$.

Proof. We have $\pi \circ \psi = 1_Q$

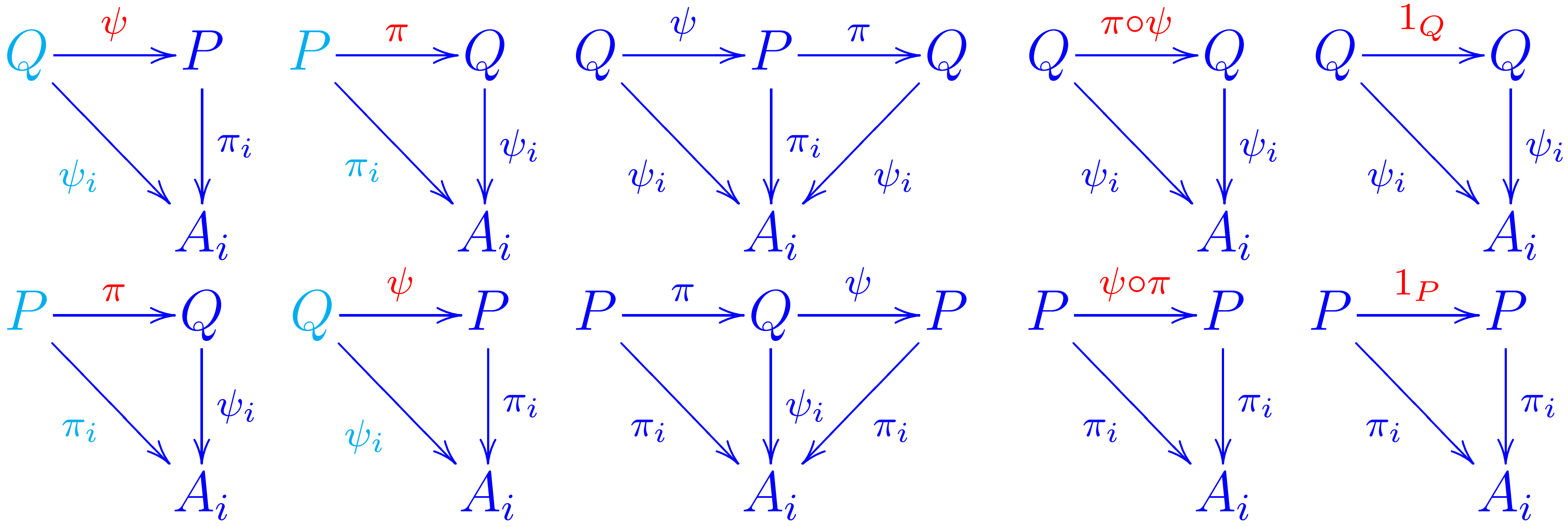


Again, it is clear that we have the commutative diagram,

Theorem (7.3)

Let $\{A_i \mid i \in I\}$ be a family of objects of a category $\mathcal{C}$. Suppose $(P, \{\pi_i\})$ and $(Q, \{\psi_i\})$ are both products of the family $\{A_i \mid i \in I\}$. Then P and Q are equivalent in $\mathcal{C}$.

Proof. We have $\pi \circ \psi = 1_Q$

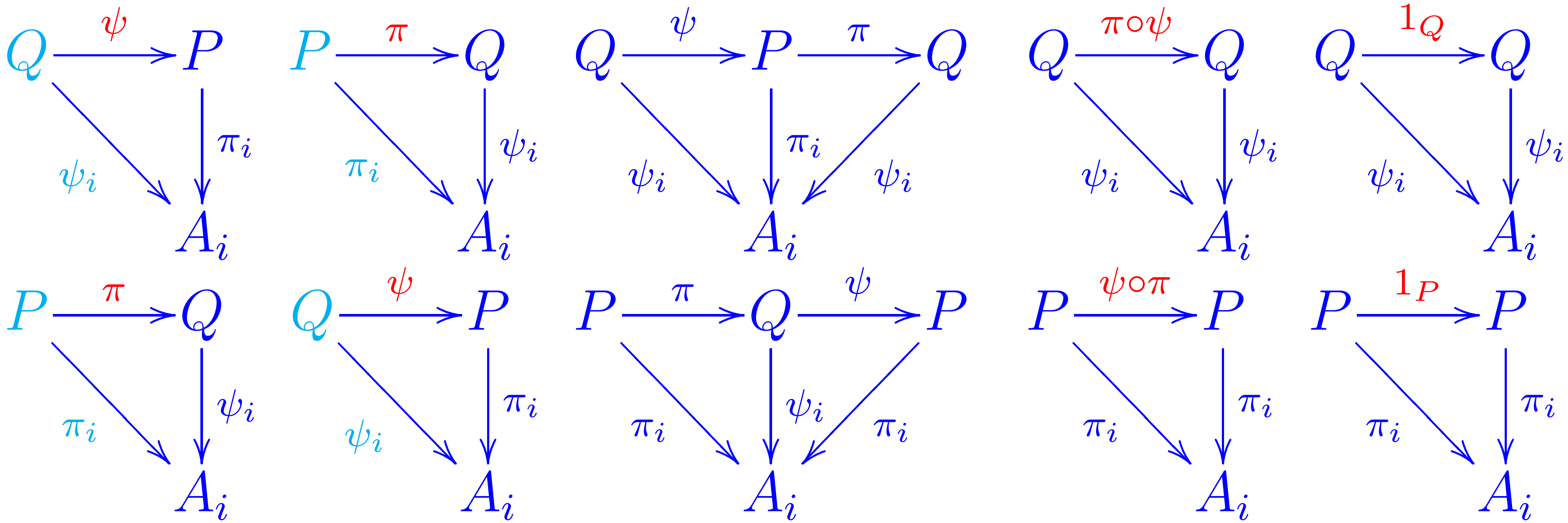


Again, it is clear that we have the commutative diagram,

Theorem (7.3)

Let $\{A_i \mid i \in I\}$ be a family of objects of a category $\mathcal{C}$. Suppose $(P, \{\pi_i\})$ and $(Q, \{\psi_i\})$ are both products of the family $\{A_i \mid i \in I\}$. Then P and Q are equivalent in $\mathcal{C}$.

Proof. We have $\pi \circ \psi = 1_Q$

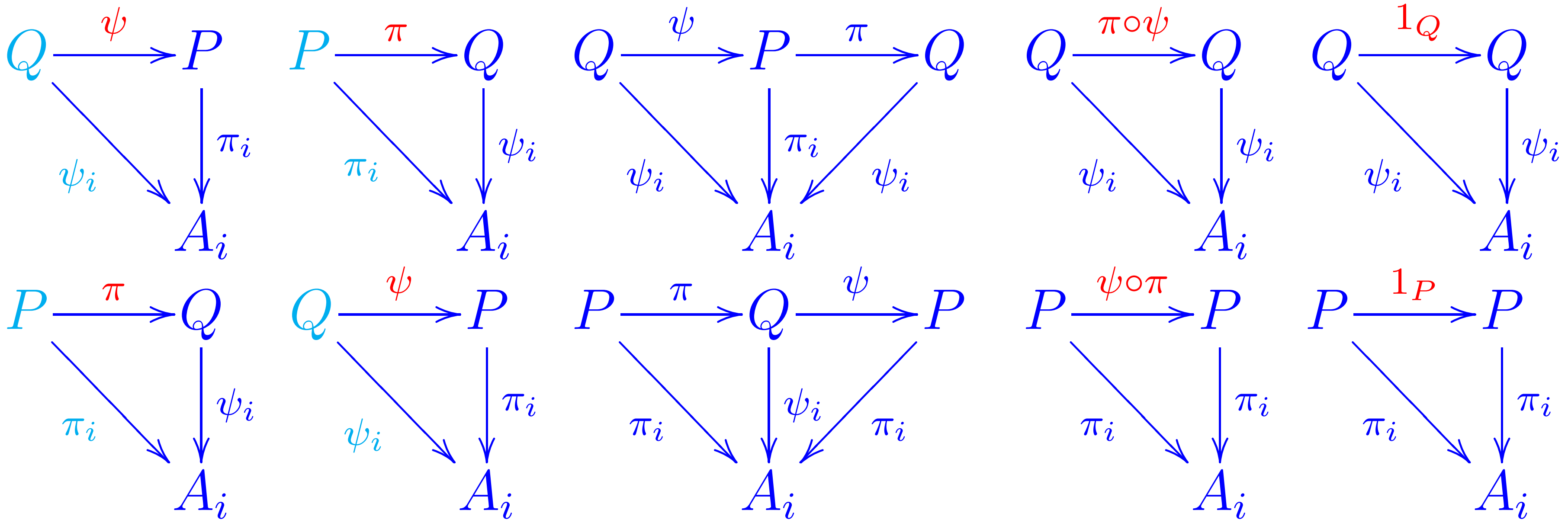


Again, it is clear that we have the commutative diagram, and by the uniqueness of the morphism $P \to P$ that makes the diagram commute,

Theorem (7.3)

Let $\{A_i \mid i \in I\}$ be a family of objects of a category $\mathcal{C}$. Suppose $(P, \{\pi_i\})$ and $(Q, \{\psi_i\})$ are both products of the family $\{A_i \mid i \in I\}$. Then P and Q are equivalent in $\mathcal{C}$.

Proof. We have $\pi \circ \psi = 1_Q$

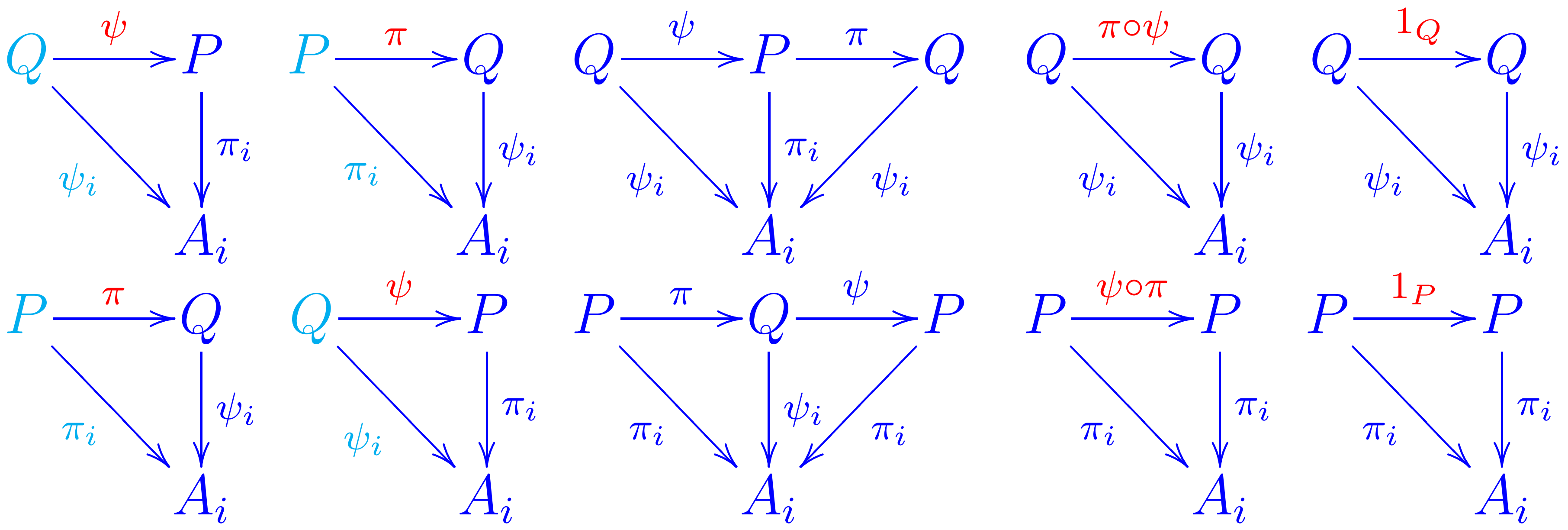


Again, it is clear that we have the commutative diagram, and by the uniqueness of the morphism $P \to P$ that makes the diagram commute, we have $\psi \circ \pi = 1_P$.

Theorem (7.3)

Let $\{A_i \mid i \in I\}$ be a family of objects of a category $\mathcal{C}$. Suppose $(P, \{\pi_i\})$ and $(Q, \{\psi_i\})$ are both products of the family $\{A_i \mid i \in I\}$. Then P and Q are equivalent in $\mathcal{C}$.

Proof. We have $\pi \circ \psi = 1_Q$ and $\psi \circ \pi = 1_P$.

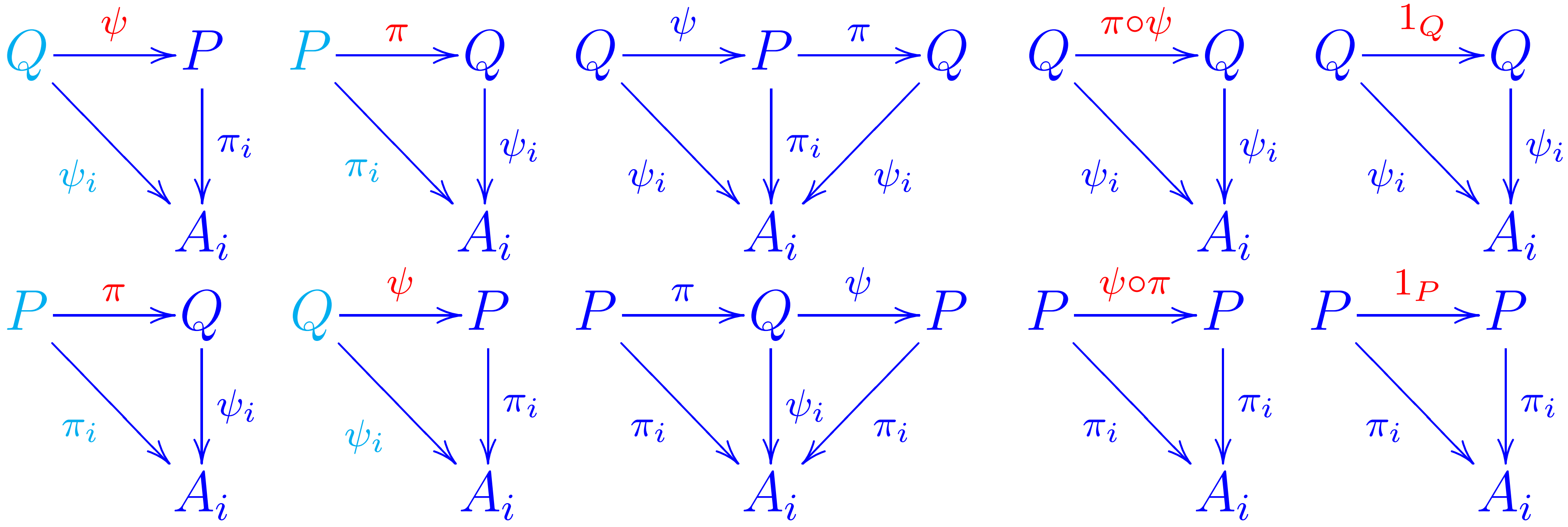


Again, it is clear that we have the commutative diagram, and by the uniqueness of the morphism $P \to P$ that makes the diagram commute, we have $\psi \circ \pi = 1_P$.

Theorem (7.3)

Let $\{A_i \mid i \in I\}$ be a family of objects of a category $\mathcal{C}$. Suppose $(P, \{\pi_i\})$ and $(Q, \{\psi_i\})$ are both products of the family $\{A_i \mid i \in I\}$. Then P and Q are equivalent in $\mathcal{C}$.

Proof. We have $\pi \circ \psi = 1_Q$ and $\psi \circ \pi = 1_P$.



Hence, P and Q are equivalent in $\mathcal{C}$.

Reversing the Arrows

Reversing the Arrows

Since abstract categories involve only objects and morphisms

Reversing the Arrows

Since abstract categories involve only objects and morphisms (no elements),

Reversing the Arrows

Since abstract categories involve only objects and morphisms (no elements), every statement about them has a dual statement,

Reversing the Arrows

Since abstract categories involve only objects and morphisms (no elements), every statement about them has a dual statement, obtained by reversing all the arrows (morphisms) in the original statement.

Reversing the Arrows

Since abstract categories involve only objects and morphisms (no elements), every statement about them has a dual statement, obtained by reversing all the arrows (morphisms) in the original statement. For example,

Reversing the Arrows

Since abstract categories involve only objects and morphisms (no elements), every statement about them has a dual statement, obtained by reversing all the arrows (morphisms) in the original statement. For example, the dual of Definition (7.2) is Definition (7.4),

Reversing the Arrows

Since abstract categories involve only objects and morphisms (no elements), every statement about them has a dual statement, obtained by reversing all the arrows (morphisms) in the original statement. For example, the dual of Definition (7.2) is Definition (7.4), and the dual of Theorem (7.3) is Theorem (7.5).

Coproducts in a Category

Coproducts in a Category

Definition (7.4). Let $\mathcal{C}$ be a category and let $\{A_i \mid i \in I\}$ be a family of objects of $\mathcal{C}$.

Coproducts in a Category

Definition (7.4). Let $\mathcal{C}$ be a category and let $\{A_i \mid i \in I\}$ be a family of objects of $\mathcal{C}$. A coproduct for the family $\{A_i \mid i \in I\}$

Coproducts in a Category

Definition (7.4). Let $\mathcal{C}$ be a category and let $\{A_i \mid i \in I\}$ be a family of objects of $\mathcal{C}$. A coproduct for the family $\{A_i \mid i \in I\}$ is an object S of $\mathcal{C}$

Coproducts in a Category

Definition (7.4). Let $\mathcal{C}$ be a category and let $\{A_i \mid i \in I\}$ be a family of objects of $\mathcal{C}$. A coproduct for the family $\{A_i \mid i \in I\}$ is an object S of $\mathcal{C}$ together with a family of morphisms $\{\iota_i : A_i \to S \mid i \in I\}$

Coproducts in a Category

Definition (7.4). Let $\mathcal{C}$ be a category and let $\{A_i \mid i \in I\}$ be a family of objects of $\mathcal{C}$. A coproduct for the family $\{A_i \mid i \in I\}$ is an object S of $\mathcal{C}$ together with a family of morphisms $\{\iota_i : A_i \to S \mid i \in I\}$ such that

Coproducts in a Category

Definition (7.4)**.** Let $\mathcal{C}$ be a category and let $\{A_i \mid i \in I\}$ be a family of objects of $\mathcal{C}$. A coproduct for the family $\{A_i \mid i \in I\}$ is an object S of $\mathcal{C}$ together with a family of morphisms $\{\iota_i : A_i \to S \mid i \in I\}$ such that for any object B

Coproducts in a Category

Definition (7.4). Let $\mathcal{C}$ be a category and let $\{A_i \mid i \in I\}$ be a family of objects of $\mathcal{C}$. A coproduct for the family $\{A_i \mid i \in I\}$ is an object S of $\mathcal{C}$ together with a family of morphisms $\{\iota_i : A_i \to S \mid i \in I\}$ such that for any object B and family of morphisms $\{\varphi_i : A_i \to B \mid i \in I\}$ in $\mathcal{C}$,

Coproducts in a Category

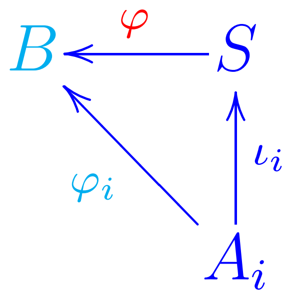
Definition (7.4). Let $\mathcal{C}$ be a category and let $\{A_i \mid i \in I\}$ be a family of objects of $\mathcal{C}$. A coproduct for the family $\{A_i \mid i \in I\}$ is an object S of $\mathcal{C}$ together with a family of morphisms $\{\iota_i : A_i \to S \mid i \in I\}$ such that for any object B and family of morphisms $\{\varphi_i : A_i \to B \mid i \in I\}$ in $\mathcal{C}$, there is a unique morphism $\varphi : S \to B$

Coproducts in a Category

Definition (7.4). Let $\mathcal{C}$ be a category and let $\{A_i \mid i \in I\}$ be a family of objects of $\mathcal{C}$. A coproduct for the family $\{A_i \mid i \in I\}$ is an object S of $\mathcal{C}$ together with a family of morphisms $\{\iota_i : A_i \to S \mid i \in I\}$ such that for any object B and family of morphisms $\{\varphi_i : A_i \to B \mid i \in I\}$ in $\mathcal{C}$, there is a unique morphism $\varphi : S \to B$ such that the diagram

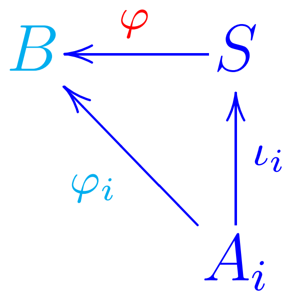
Coproducts in a Category

Definition (7.4). Let $\mathcal{C}$ be a category and let $\{A_i \mid i \in I\}$ be a family of objects of $\mathcal{C}$. A coproduct for the family $\{A_i \mid i \in I\}$ is an object S of $\mathcal{C}$ together with a family of morphisms $\{\iota_i : A_i \to S \mid i \in I\}$ such that for any object B and family of morphisms $\{\varphi_i : A_i \to B \mid i \in I\}$ in $\mathcal{C}$, there is a unique morphism $\varphi : S \to B$ such that the diagram



Coproducts in a Category

Definition (7.4). Let $\mathcal{C}$ be a category and let $\{A_i \mid i \in I\}$ be a family of objects of $\mathcal{C}$. A coproduct for the family $\{A_i \mid i \in I\}$ is an object S of $\mathcal{C}$ together with a family of morphisms $\{\iota_i : A_i \to S \mid i \in I\}$ such that for any object B and family of morphisms $\{\varphi_i : A_i \to B \mid i \in I\}$ in $\mathcal{C}$, there is a unique morphism $\varphi : S \to B$ such that the diagram



commutes for all $i \in I$.

Reversing the Arrows

Reversing the Arrows

By reversing the arrows in the proof of the following theorem for products,

Reversing the Arrows

By reversing the arrows in the proof of the following theorem for products,

Theorem (7.3)**.** Let $\{A_i \mid i \in I\}$ be a family of objects of a category $\mathcal{C}$.

Reversing the Arrows

By reversing the arrows in the proof of the following theorem for products,

Theorem (7.3)**.** Let $\{A_i \mid i \in I\}$ be a family of objects of a category $\mathcal{C}$. Suppose $(P, \{\pi_i\})$ and $(Q, \{\psi_i\})$ are both products of the family $\{A_i \mid i \in I\}$.

Reversing the Arrows

By reversing the arrows in the proof of the following theorem for products,

Theorem (7.3)**.** Let $\{A_i \mid i \in I\}$ be a family of objects of a category $\mathcal{C}$. Suppose $(P, \{\pi_i\})$ and $(Q, \{\psi_i\})$ are both products of the family $\{A_i \mid i \in I\}$. Then P and Q are equivalent in $\mathcal{C}$.

Reversing the Arrows

By reversing the arrows in the proof of the following theorem for products,

Theorem (7.3). Let $\{A_i \mid i \in I\}$ be a family of objects of a category $\mathcal{C}$. Suppose $(P, \{\pi_i\})$ and $(Q, \{\psi_i\})$ are both products of the family $\{A_i \mid i \in I\}$. Then P and Q are equivalent in $\mathcal{C}$.

we can show the following theorem for coproducts.

Reversing the Arrows

By reversing the arrows in the proof of the following theorem for products,

Theorem (7.3). Let $\{A_i \mid i \in I\}$ be a family of objects of a category $\mathcal{C}$. Suppose $(P, \{\pi_i\})$ and $(Q, \{\psi_i\})$ are both products of the family $\{A_i \mid i \in I\}$. Then P and Q are equivalent in $\mathcal{C}$.

we can show the following theorem for coproducts.

Theorem (7.5). Let $\{A_i \mid i \in I\}$ be a family of objects of a category $\mathcal{C}$.

Reversing the Arrows

By reversing the arrows in the proof of the following theorem for products,

Theorem (7.3)**.** Let $\{A_i \mid i \in I\}$ be a family of objects of a category $\mathcal{C}$. Suppose $(P, \{\pi_i\})$ and $(Q, \{\psi_i\})$ are both products of the family $\{A_i \mid i \in I\}$. Then P and Q are equivalent in $\mathcal{C}$.

we can show the following theorem for coproducts.

Theorem (7.5)**.** Let $\{A_i \mid i \in I\}$ be a family of objects of a category $\mathcal{C}$. Suppose $(S, \{\iota_i\})$ and $(T, \{\lambda_i\})$ are both coproducts of the family $\{A_i \mid i \in I\}$.

Reversing the Arrows

By reversing the arrows in the proof of the following theorem for products,

Theorem (7.3)**.** Let $\{A_i \mid i \in I\}$ be a family of objects of a category $\mathcal{C}$. Suppose $(P, \{\pi_i\})$ and $(Q, \{\psi_i\})$ are both products of the family $\{A_i \mid i \in I\}$. Then P and Q are equivalent in $\mathcal{C}$.

we can show the following theorem for coproducts.

Theorem (7.5)**.** Let $\{A_i \mid i \in I\}$ be a family of objects of a category $\mathcal{C}$. Suppose $(S, \{\iota_i\})$ and $(T, \{\lambda_i\})$ are both coproducts of the family $\{A_i \mid i \in I\}$. Then S and T are equivalent in $\mathcal{C}$.

Concrete Categories

Concrete Categories

Roughly speaking,

Concrete Categories

Roughly speaking, a concrete category $\mathcal{C}$ is a category

Concrete Categories

Roughly speaking, a concrete category $\mathcal{C}$ is a category whose objects are some special kind of sets

Concrete Categories

Roughly speaking, a concrete category $\mathcal{C}$ is a category whose objects are some special kind of sets and whose morphisms are some special kind of maps,

Concrete Categories

Roughly speaking, a concrete category $\mathcal{C}$ is a category whose objects are some special kind of sets and whose morphisms are some special kind of maps, and the composite of two morphisms in $\mathcal{C}$ is the composition of these two maps,

Concrete Categories

Roughly speaking, a concrete category $\mathcal{C}$ is a category whose objects are some special kind of sets and whose morphisms are some special kind of maps, and the composite of two morphisms in $\mathcal{C}$ is the composition of these two maps, and the identity morphisms are the identity maps.

Concrete Categories

Roughly speaking, a concrete category $\mathcal{C}$ is a category whose objects are some special kind of sets and whose morphisms are some special kind of maps, and the composite of two morphisms in $\mathcal{C}$ is the composition of these two maps, and the identity morphisms are the identity maps.

For example,

Concrete Categories

Roughly speaking, a concrete category $\mathcal{C}$ is a category whose objects are some special kind of sets and whose morphisms are some special kind of maps, and the composite of two morphisms in $\mathcal{C}$ is the composition of these two maps, and the identity morphisms are the identity maps.

For example, the categories $\mathcal{S}$, $\mathcal{G}$, $\mathcal{A}$, and $\mathcal{T}$ are all concrete categories.

Concrete Categories

Roughly speaking, a concrete category $\mathcal{C}$ is a category whose objects are some special kind of sets and whose morphisms are some special kind of maps, and the composite of two morphisms in $\mathcal{C}$ is the composition of these two maps, and the identity morphisms are the identity maps.

For example, the categories $\mathcal{S}$, $\mathcal{G}$, $\mathcal{A}$, and $\mathcal{T}$ are all concrete categories.

Concrete Categories

Roughly speaking, a concrete category $\mathcal{C}$ is a category whose objects are some special kind of sets and whose morphisms are some special kind of maps, and the composite of two morphisms in $\mathcal{C}$ is the composition of these two maps, and the identity morphisms are the identity maps.

For example, the categories $\mathcal{S}$, $\mathcal{G}$, $\mathcal{A}$, and $\mathcal{T}$ are all concrete categories.

Concrete Categories

Roughly speaking, a concrete category $\mathcal{C}$ is a category whose objects are some special kind of sets and whose morphisms are some special kind of maps, and the composite of two morphisms in $\mathcal{C}$ is the composition of these two maps, and the identity morphisms are the identity maps.

For example, the categories $\mathcal{S}$, $\mathcal{G}$, $\mathcal{A}$, and $\mathcal{T}$ are all concrete categories.

Concrete Categories

Roughly speaking, a concrete category $\mathcal{C}$ is a category whose objects are some special kind of sets and whose morphisms are some special kind of maps, and the composite of two morphisms in $\mathcal{C}$ is the composition of these two maps, and the identity morphisms are the identity maps.

For example, the categories $\mathcal{S}$, $\mathcal{G}$, $\mathcal{A}$, and $\mathcal{T}$ are all concrete categories.

Concrete Categories

Roughly speaking, a concrete category $\mathcal{C}$ is a category whose objects are some special kind of sets and whose morphisms are some special kind of maps, and the composite of two morphisms in $\mathcal{C}$ is the composition of these two maps, and the identity morphisms are the identity maps.

For example, the categories $\mathcal{S}$, $\mathcal{G}$, $\mathcal{A}$, and $\mathcal{T}$ are all concrete categories.

In some sense, a concrete category

Concrete Categories

Roughly speaking, a concrete category $\mathcal{C}$ is a category whose objects are some special kind of sets and whose morphisms are some special kind of maps, and the composite of two morphisms in $\mathcal{C}$ is the composition of these two maps, and the identity morphisms are the identity maps.

For example, the categories $\mathcal{S}$, $\mathcal{G}$, $\mathcal{A}$, and $\mathcal{T}$ are all concrete categories.

In some sense, a concrete category is just a "sub-category" of the category $\mathcal{S}$ of sets.

Free Objects

Free Objects

Definition (7.7)**.** Let F be an object in a concrete category $\mathcal{C}$,

Free Objects

Definition (7.7). Let F be an object in a concrete category $\mathcal{C}$, let X be a nonempty set,

Free Objects

Definition (7.7)**.** Let F be an object in a concrete category $\mathcal{C}$, let X be a nonempty set, and let $i : X \to F$ be a map (of sets).

Free Objects

Definition (7.7)**.** Let F be an object in a concrete category $\mathcal{C}$, let X be a nonempty set, and let $i : X \to F$ be a map (of sets). F is said to be free on the set X

Free Objects

Definition (7.7)**.** Let F be an object in a concrete category $\mathcal{C}$, let X be a nonempty set, and let $i : X \longrightarrow F$ be a map (of sets). F is said to be free on the set X if for any object A of $\mathcal{C}$

Free Objects

Definition (7.7)**.** Let F be an object in a concrete category $\mathcal{C}$, let X be a nonempty set, and let $i : X \to F$ be a map (of sets). F is said to be free on the set X if for any object A of $\mathcal{C}$ and map (of sets) $f : X \to A$,

Free Objects

Definition (7.7). Let F be an object in a concrete category $\mathcal{C}$, let X be a nonempty set, and let $i : X \to F$ be a map (of sets). F is said to be free on the set X if for any object A of $\mathcal{C}$ and map (of sets) $f : X \to A$, there exists a unique morphism $\bar{f} : F \to A$ of $\mathcal{C}$

Free Objects

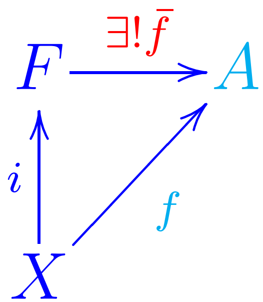
Definition (7.7)**.** Let F be an object in a concrete category $\mathcal{C}$, let X be a nonempty set, and let $i : X \to F$ be a map (of sets). F is said to be free on the set X if for any object A of $\mathcal{C}$ and map (of sets) $f : X \to A$, there exists a unique morphism $\bar{f} : F \to A$ of $\mathcal{C}$ such that $\bar{f}i = f$

Free Objects

Definition (7.7)**.** Let F be an object in a concrete category $\mathcal{C}$, let X be a nonempty set, and let $i : X \to F$ be a map (of sets). F is said to be free on the set X if for any object A of $\mathcal{C}$ and map (of sets) $f : X \to A$, there exists a unique morphism $\bar{f} : F \to A$ of $\mathcal{C}$ such that $\bar{f}i = f$ (as a map of sets $X \to A$),

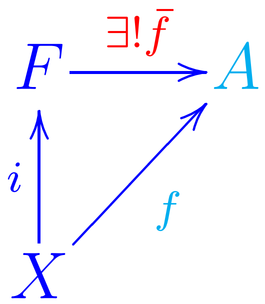
Free Objects

Definition (7.7). Let F be an object in a concrete category $\mathcal{C}$, let X be a nonempty set, and let $i : X \to F$ be a map (of sets). F is said to be free on the set X if for any object A of $\mathcal{C}$ and map (of sets) $f : X \to A$, there exists a unique morphism $\bar{f} : F \to A$ of $\mathcal{C}$ such that $\bar{f}i = f$ (as a map of sets $X \to A$), i.e.,



Free Objects

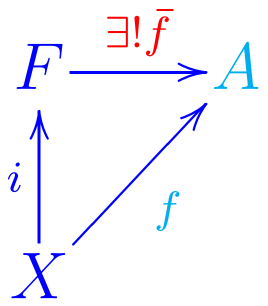
Definition (7.7)**.** Let F be an object in a concrete category $\mathcal{C}$, let X be a nonempty set, and let $i : X \to F$ be a map (of sets). F is said to be free on the set X if for any object A of $\mathcal{C}$ and map (of sets) $f : X \to A$, there exists a unique morphism $\bar{f} : F \to A$ of $\mathcal{C}$ such that $\bar{f}i = f$ (as a map of sets $X \to A$), i.e.,



Remark. The essential fact about a free object F is that

Free Objects

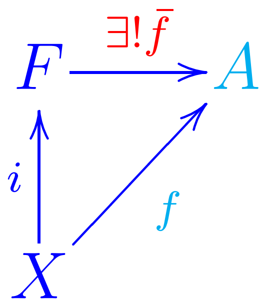
Definition (7.7)**.** Let F be an object in a concrete category $\mathcal{C}$, let X be a nonempty set, and let $i : X \to F$ be a map (of sets). F is said to be free on the set X if for any object A of $\mathcal{C}$ and map (of sets) $f : X \to A$, there exists a unique morphism $\bar{f} : F \to A$ of $\mathcal{C}$ such that $\bar{f}i = f$ (as a map of sets $X \to A$), i.e.,



Remark. The essential fact about a free object F is that in order to define a morphism with domain F,

Free Objects

Definition (7.7)**.** Let F be an object in a concrete category $\mathcal{C}$, let X be a nonempty set, and let $i : X \to F$ be a map (of sets). F is said to be free on the set X if for any object A of $\mathcal{C}$ and map (of sets) $f : X \to A$, there exists a unique morphism $\bar{f} : F \to A$ of $\mathcal{C}$ such that $\bar{f}i = f$ (as a map of sets $X \to A$), i.e.,



Remark. The essential fact about a free object F is that in order to define a morphism with domain F, it suffices to specify the image of the subset $i(X)$.

A Familiar Example

A Familiar Example

Example. Consider the category $\mathcal{V}$,

A Familiar Example

Example. Consider the category $\mathcal{V}$, whose objects are all vector spaces over $\mathbb{R}$

A Familiar Example

Example. Consider the category $\mathcal{V}$, whose objects are all vector spaces over $\mathbb{R}$ and whose morphisms are linear transformations.

A Familiar Example

Example. Consider the category $\mathcal{V}$, whose objects are all vector spaces over $\mathbb{R}$ and whose morphisms are linear transformations. Then $\mathcal{V}$ is a concrete category.

A Familiar Example

Example. Consider the category $\mathcal{V}$, whose objects are all vector spaces over $\mathbb{R}$ and whose morphisms are linear transformations. Then $\mathcal{V}$ is a concrete category. Note that if B is a basis for a vector space V,

A Familiar Example

Example. Consider the category $\mathcal{V}$, whose objects are all vector spaces over $\mathbb{R}$ and whose morphisms are linear transformations. Then $\mathcal{V}$ is a concrete category. Note that if B is a basis for a vector space V, then any linear transformation $T : V \to W$

A Familiar Example

Example. Consider the category $\mathcal{V}$, whose objects are all vector spaces over $\mathbb{R}$ and whose morphisms are linear transformations. Then $\mathcal{V}$ is a concrete category. Note that if B is a basis for a vector space V, then any linear transformation $T : V \to W$ is completely determined by $T(B)$;

A Familiar Example

Example. Consider the category $\mathcal{V}$, whose objects are all vector spaces over $\mathbb{R}$ and whose morphisms are linear transformations. Then $\mathcal{V}$ is a concrete category. Note that if B is a basis for a vector space V, then any linear transformation $T : V \to W$ is completely determined by $T(B)$; in other words,

A Familiar Example

Example. Consider the category $\mathcal{V}$, whose objects are all vector spaces over $\mathbb{R}$ and whose morphisms are linear transformations. Then $\mathcal{V}$ is a concrete category. Note that if B is a basis for a vector space V, then any linear transformation $T : V \to W$ is completely determined by $T(B)$; in other words, for any vector space W and map $f : B \to W$

A Familiar Example

Example. Consider the category $\mathcal{V}$, whose objects are all vector spaces over $\mathbb{R}$ and whose morphisms are linear transformations. Then $\mathcal{V}$ is a concrete category. Note that if B is a basis for a vector space V, then any linear transformation $T : V \to W$ is completely determined by $T(B)$; in other words, for any vector space W and map $f : B \to W$ there exists a unique linear transformation $T : V \to W$ such that $T(v) = f(v)\ \forall v \in B$

A Familiar Example

Example. Consider the category $\mathcal{V}$, whose objects are all vector spaces over $\mathbb{R}$ and whose morphisms are linear transformations. Then $\mathcal{V}$ is a concrete category. Note that if B is a basis for a vector space V, then any linear transformation $T : V \to W$ is completely determined by $T(B)$; in other words, for any vector space W and map $f : B \to W$ there exists a unique linear transformation $T : V \to W$ such that $T(v) = f(v)\ \forall v \in B$ i.e.,

$$\begin{array}{ccc} V & \xrightarrow{\exists!\ T} & W \\ \uparrow i & \nearrow f & \\ B & & \end{array}$$

A Familiar Example

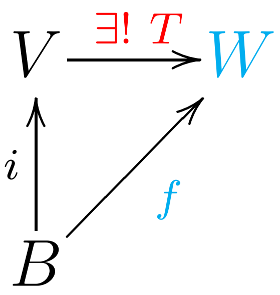
Example. Consider the category $\mathcal{V}$, whose objects are all vector spaces over $\mathbb{R}$ and whose morphisms are linear transformations. Then $\mathcal{V}$ is a concrete category. Note that if B is a basis for a vector space V, then any linear transformation $T : V \to W$ is completely determined by $T(B)$; in other words, for any vector space W and map $f : B \to W$ there exists a unique linear transformation $T : V \to W$ such that $T(v) = f(v)\ \forall v \in B$ i.e.,

$$\begin{array}{ccc} V & \xrightarrow{\exists!\ T} & W \\ {\scriptstyle i}\uparrow & \nearrow_{f} & \\ B & & \end{array}$$

where $i : B \to V$ is the inclusion map.

A Familiar Example

Example. Consider the category $\mathcal{V}$, whose objects are all vector spaces over $\mathbb{R}$ and whose morphisms are linear transformations. Then $\mathcal{V}$ is a concrete category. Note that if B is a basis for a vector space V, then any linear transformation $T : V \to W$ is completely determined by $T(B)$; in other words, for any vector space W and map $f : B \to W$ there exists a unique linear transformation $T : V \to W$ such that $T(v) = f(v)\ \forall v \in B$ i.e.,

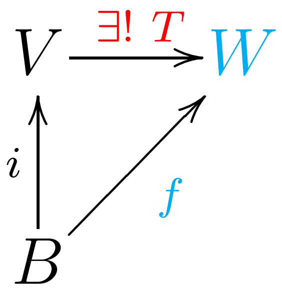


where $i : B \to V$ is the inclusion map.

Hence, V is free on the set B in the category $\mathcal{V}$.

A Familiar Example

Example. Consider the category $\mathcal{V}$, whose objects are all vector spaces over $\mathbb{R}$ and whose morphisms are linear transformations. Then $\mathcal{V}$ is a concrete category. Note that if B is a basis for a vector space V, then any linear transformation $T : V \to W$ is completely determined by $T(B)$; in other words, for any vector space W and map $f : B \to W$ there exists a unique linear transformation $T : V \to W$ such that $T(v) = f(v)\ \forall v \in B$ i.e.,



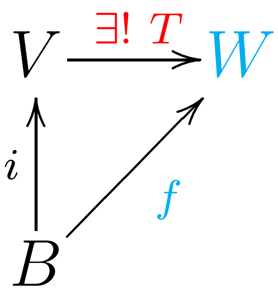
where $i : B \to V$ is the inclusion map.

Hence, V is free on the set B in the category $\mathcal{V}$.

In this catecory $\mathcal{V}$,

A Familiar Example

Example. Consider the category $\mathcal{V}$, whose objects are all vector spaces over $\mathbb{R}$ and whose morphisms are linear transformations. Then $\mathcal{V}$ is a concrete category. Note that if B is a basis for a vector space V, then any linear transformation $T : V \to W$ is completely determined by $T(B)$; in other words, for any vector space W and map $f : B \to W$ there exists a unique linear transformation $T : V \to W$ such that $T(v) = f(v)\ \forall v \in B$ i.e.,



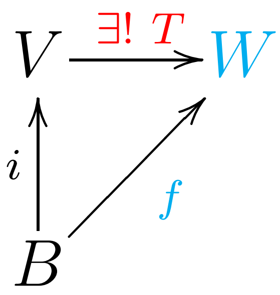
where $i : B \to V$ is the inclusion map.

Hence, V is free on the set B in the category $\mathcal{V}$.

In this catecory $\mathcal{V}$, except the object $\{0\}$,

A Familiar Example

Example. Consider the category $\mathcal{V}$, whose objects are all vector spaces over $\mathbb{R}$ and whose morphisms are linear transformations. Then $\mathcal{V}$ is a concrete category. Note that if B is a basis for a vector space V, then any linear transformation $T : V \to W$ is completely determined by $T(B)$; in other words, for any vector space W and map $f : B \to W$ there exists a unique linear transformation $T : V \to W$ such that $T(v) = f(v)\ \forall v \in B$ i.e.,



where $i : B \to V$ is the inclusion map.

Hence, V is free on the set B in the category $\mathcal{V}$.

In this catecory $\mathcal{V}$, except the object $\{0\}$, every object is a free object.

A Familiar Result

Remark. Let $\mathcal{V}$ be the category of vector spaces over $\mathbb{R}$.

A Familiar Result

Remark. Let $\mathcal{V}$ be the category of vector spaces over $\mathbb{R}$.

- A morphism T of $\mathcal{V}$ is an equivalence

A Familiar Result

Remark. Let $\mathcal{V}$ be the category of vector spaces over $\mathbb{R}$.

- A morphism T of $\mathcal{V}$ is an equivalence if and only if T is an isomorphism.

A Familiar Result

Remark. Let $\mathcal{V}$ be the category of vector spaces over $\mathbb{R}$.

- A morphism T of $\mathcal{V}$ is an equivalence if and only if T is an isomorphism.
- If two vector spaces have the same dimensions,

A Familiar Result

Remark. Let $\mathcal{V}$ be the category of vector spaces over $\mathbb{R}$.

- A morphism T of $\mathcal{V}$ is an equivalence if and only if T is an isomorphism.
- If two vector spaces have the same dimensions, i.e., their bases have the same cardinality,

A Familiar Result

Remark. Let $\mathcal{V}$ be the category of vector spaces over $\mathbb{R}$.

- A morphism T of $\mathcal{V}$ is an equivalence if and only if T is an isomorphism.
- If two vector spaces have the same dimensions, i.e., their bases have the same cardinality, then these two vector spaces are isomorphic,

A Familiar Result

Remark. Let $\mathcal{V}$ be the category of vector spaces over $\mathbb{R}$.

- A morphism T of $\mathcal{V}$ is an equivalence if and only if T is an isomorphism.
- If two vector spaces have the same dimensions, i.e., their bases have the same cardinality, then these two vector spaces are isomorphic, i.e., they are equivalent in the category $\mathcal{V}$.

A Familiar Result

Remark. Let $\mathcal{V}$ be the category of vector spaces over $\mathbb{R}$.

- A morphism T of $\mathcal{V}$ is an equivalence if and only if T is an isomorphism.
- If two vector spaces have the same dimensions, i.e., their bases have the same cardinality, then these two vector spaces are isomorphic, i.e., they are equivalent in the category $\mathcal{V}$.
  The following theorem is a similar result for free objects.

A Familiar Result

Remark. Let $\mathcal{V}$ be the category of vector spaces over $\mathbb{R}$.

- A morphism T of $\mathcal{V}$ is an equivalence if and only if T is an isomorphism.
- If two vector spaces have the same dimensions, i.e., their bases have the same cardinality, then these two vector spaces are isomorphic, i.e., they are equivalent in the category $\mathcal{V}$.
  The following theorem is a similar result for free objects.

Theorem (7.8)**.** Let F and F' be objects in a concrete category $\mathcal{C}$.

A Familiar Result

Remark. Let $\mathcal{V}$ be the category of vector spaces over $\mathbb{R}$.

- A morphism T of $\mathcal{V}$ is an equivalence if and only if T is an isomorphism.
- If two vector spaces have the same dimensions, i.e., their bases have the same cardinality, then these two vector spaces are isomorphic, i.e., they are equivalent in the category $\mathcal{V}$.
  The following theorem is a similar result for free objects.

Theorem (7.8)**.** Let F and F' be objects in a concrete category $\mathcal{C}$. Suppose that F is free on the set X and F' is free on the set X'.

A Familiar Result

Remark. Let $\mathcal{V}$ be the category of vector spaces over $\mathbb{R}$.

- A morphism T of $\mathcal{V}$ is an equivalence if and only if T is an isomorphism.
- If two vector spaces have the same dimensions, i.e., their bases have the same cardinality, then these two vector spaces are isomorphic, i.e., they are equivalent in the category $\mathcal{V}$.
  The following theorem is a similar result for free objects.

Theorem (7.8). Let F and F' be objects in a concrete category $\mathcal{C}$. Suppose that F is free on the set X and F' is free on the set X'. If $|X| = |X'|$,

A Familiar Result

Remark. Let $\mathcal{V}$ be the category of vector spaces over $\mathbb{R}$.

- A morphism T of $\mathcal{V}$ is an equivalence if and only if T is an isomorphism.
- If two vector spaces have the same dimensions, i.e., their bases have the same cardinality, then these two vector spaces are isomorphic, i.e., they are equivalent in the category $\mathcal{V}$.
  The following theorem is a similar result for free objects.

Theorem (7.8). Let F and F' be objects in a concrete category $\mathcal{C}$. Suppose that F is free on the set X and F' is free on the set X'. If $|X| = |X'|$, then F and F' are equivalent.

Theorem (7.8)

Let F and F' be objects in a concrete category $\mathcal{C}$. Suppose that F is free on the set X and F' is free on the set X'. If $|X| = |X'|$, then F and F' are equivalent.

Theorem (7.8)

Let F and F' be objects in a concrete category $\mathcal{C}$. Suppose that F is free on the set X and F' is free on the set X'. If $|X| = |X'|$, then F and F' are equivalent.

You won't have time to take notes for this proof.

Theorem (7.8)

Let F and F' be objects in a concrete category $\mathcal{C}$. Suppose that F is free on the set X and F' is free on the set X'. If $|X| = |X'|$, then F and F' are equivalent.

You won't have time to take notes for this proof. Please put your pens down and pay attention to my presentation.

Theorem (7.8)

Let F and F' be objects in a concrete category $\mathcal{C}$. Suppose that F is free on the set X and F' is free on the set X'. If $|X| = |X'|$, then F and F' are equivalent.

You won't have time to take notes for this proof. Please put your pens down and pay attention to my presentation. If you decide to take notes,

Theorem (7.8)

Let F and F' be objects in a concrete category $\mathcal{C}$. Suppose that F is free on the set X and F' is free on the set X'. If $|X| = |X'|$, then F and F' are equivalent.

You won't have time to take notes for this proof. Please put your pens down and pay attention to my presentation. If you decide to take notes, you may sacrifice understanding the proof.

Theorem (7.8)

Let F and F' be objects in a concrete category $\mathcal{C}$. Suppose that F is free on the set X and F' is free on the set X'. If $|X| = |X'|$, then F and F' are equivalent.

Proof. Since $|X| = |X'|$,

Theorem (7.8)

Let F and F' be objects in a concrete category $\mathcal{C}$. Suppose that F is free on the set X and F' is free on the set X'. If $|X| = |X'|$, then F and F' are equivalent.

Proof. Since $|X| = |X'|$, there is a bijection $f : X \to X'$

Theorem (7.8)

Let F and F' be objects in a concrete category $\mathcal{C}$. Suppose that F is free on the set X and F' is free on the set X'. If $|X| = |X'|$, then F and F' are equivalent.

Proof. Since $|X| = |X'|$, there is a bijection $f : X \to X'$ and hence $f^{-1} : X' \to X$ exists.

Theorem (7.8)

Let F and F' be objects in a concrete category $\mathcal{C}$. Suppose that F is free on the set X and F' is free on the set X'. If $|X| = |X'|$, then F and F' are equivalent.

Proof. Since $|X| = |X'|$, there is a bijection $f : X \to X'$ and hence $f^{-1} : X' \to X$ exists. Let $i : X \to F$ and $i' : X' \to F'$ be the corresponding maps in the definition,

Theorem (7.8)

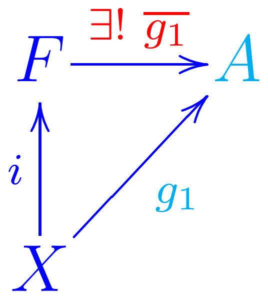
Let F and F' be objects in a concrete category $\mathcal{C}$. Suppose that F is free on the set X and F' is free on the set X'. If $|X| = |X'|$, then F and F' are equivalent.

Proof. Since $|X| = |X'|$, there is a bijection $f : X \to X'$ and hence $f^{-1} : X' \to X$ exists. Let $i : X \to F$ and $i' : X' \to F'$ be the corresponding maps in the definition, i.e.,

Theorem (7.8)

Let F and F' be objects in a concrete category $\mathcal{C}$. Suppose that F is free on the set X and F' is free on the set X'. If $|X| = |X'|$, then F and F' are equivalent.

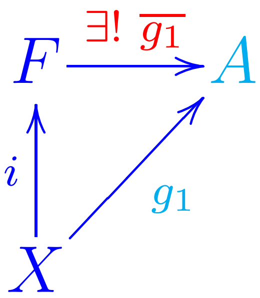
Proof. Since $|X| = |X'|$, there is a bijection $f : X \to X'$ and hence $f^{-1} : X' \to X$ exists. Let $i : X \to F$ and $i' : X' \to F'$ be the corresponding maps in the definition, i.e.,



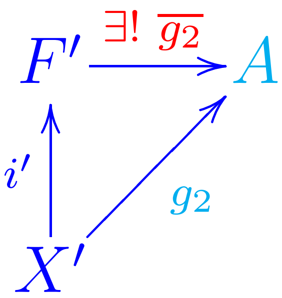
Theorem (7.8)

Let F and F' be objects in a concrete category $\mathcal{C}$. Suppose that F is free on the set X and F' is free on the set X'. If $|X| = |X'|$, then F and F' are equivalent.

Proof. Since $|X| = |X'|$, there is a bijection $f : X \to X'$ and hence $f^{-1} : X' \to X$ exists. Let $i : X \to F$ and $i' : X' \to F'$ be the corresponding maps in the definition, i.e.,



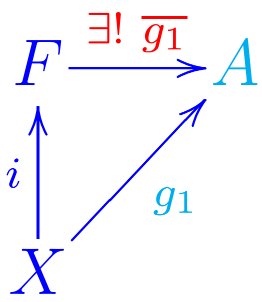
and



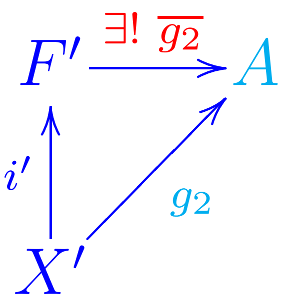
Theorem (7.8)

Let F and F' be objects in a concrete category $\mathcal{C}$. Suppose that F is free on the set X and F' is free on the set X'. If $|X| = |X'|$, then F and F' are equivalent.

Proof. Since $|X| = |X'|$, there is a bijection $f : X \to X'$ and hence $f^{-1} : X' \to X$ exists. Let $i : X \to F$ and $i' : X' \to F'$ be the corresponding maps in the definition, i.e.,



and

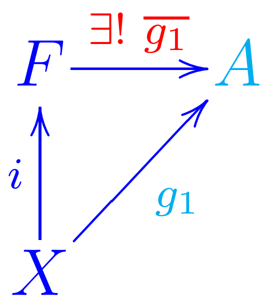


Take $g_1 = i' \circ f : X \to X' \to F'$

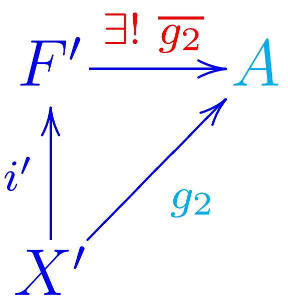
Theorem (7.8)

Let F and F' be objects in a concrete category $\mathcal{C}$. Suppose that F is free on the set X and F' is free on the set X'. If $|X| = |X'|$, then F and F' are equivalent.

Proof. Since $|X| = |X'|$, there is a bijection $f : X \to X'$ and hence $f^{-1} : X' \to X$ exists. Let $i : X \to F$ and $i' : X' \to F'$ be the corresponding maps in the definition, i.e.,



and

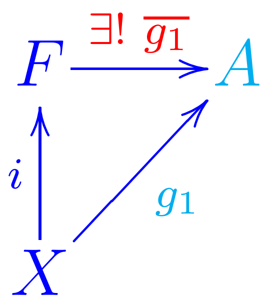


Take $g_1 = i' \circ f : X \to X' \to F'$ and
$g_2 = i \circ f^{-1} : X' \to X \to F$.

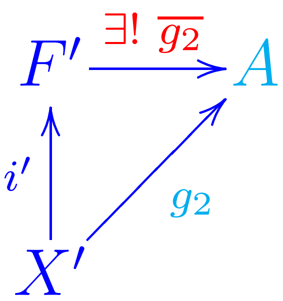
Theorem (7.8)

Let F and F' be objects in a concrete category $\mathcal{C}$. Suppose that F is free on the set X and F' is free on the set X'. If $|X| = |X'|$, then F and F' are equivalent.

Proof. Since $|X| = |X'|$, there is a bijection $f : X \to X'$ and hence $f^{-1} : X' \to X$ exists. Let $i : X \to F$ and $i' : X' \to F'$ be the corresponding maps in the definition, i.e.,



and

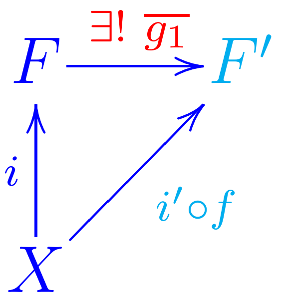


Take $g_1 = i' \circ f : X \to X' \to F'$ and
$g_2 = i \circ f^{-1} : X' \to X \to F$. Then we have

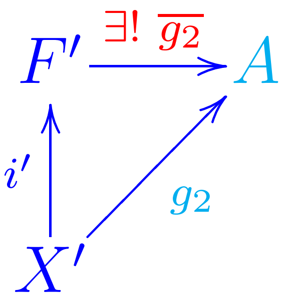
Theorem (7.8)

Let F and F' be objects in a concrete category $\mathcal{C}$. Suppose that F is free on the set X and F' is free on the set X'. If $|X| = |X'|$, then F and F' are equivalent.

Proof. Since $|X| = |X'|$, there is a bijection $f : X \to X'$ and hence $f^{-1} : X' \to X$ exists. Let $i : X \to F$ and $i' : X' \to F'$ be the corresponding maps in the definition, i.e.,



and

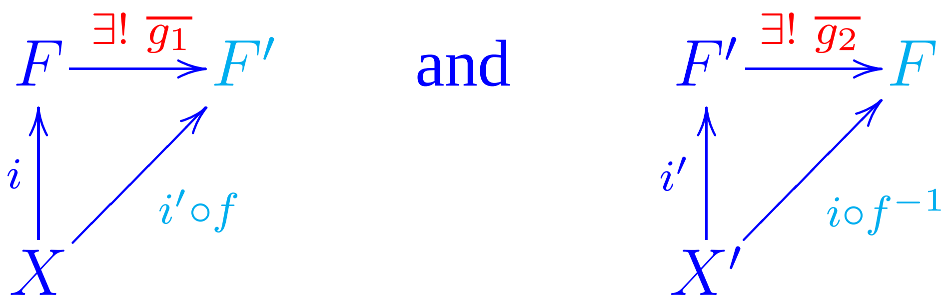


Take $g_1 = i' \circ f : X \to X' \to F'$ and $g_2 = i \circ f^{-1} : X' \to X \to F$. Then we have

Theorem (7.8)

Let F and F' be objects in a concrete category $\mathcal{C}$. Suppose that F is free on the set X and F' is free on the set X'. If $|X| = |X'|$, then F and F' are equivalent.

Proof. Since $|X| = |X'|$, there is a bijection $f : X \to X'$ and hence $f^{-1} : X' \to X$ exists. Let $i : X \to F$ and $i' : X' \to F'$ be the corresponding maps in the definition, i.e.,

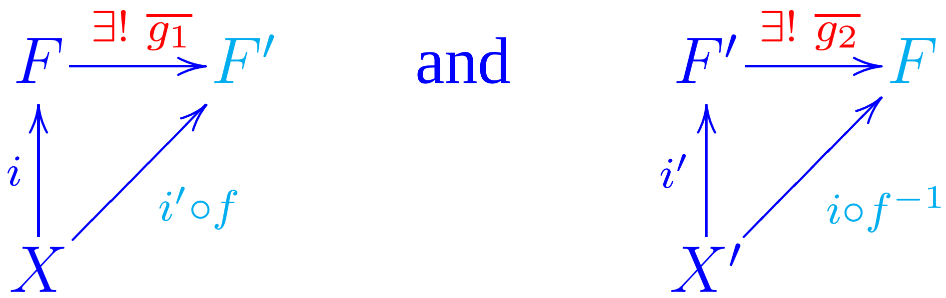


Take $g_1 = i' \circ f : X \to X' \to F'$ and
$g_2 = i \circ f^{-1} : X' \to X \to F$. Then we have

Theorem (7.8)

Let F and F' be objects in a concrete category $\mathcal{C}$. Suppose that F is free on the set X and F' is free on the set X'. If $|X| = |X'|$, then F and F' are equivalent.

Proof. Since $|X| = |X'|$, there is a bijection $f : X \to X'$ and hence $f^{-1} : X' \to X$ exists. Let $i : X \to F$ and $i' : X' \to F'$ be the corresponding maps in the definition, i.e.,

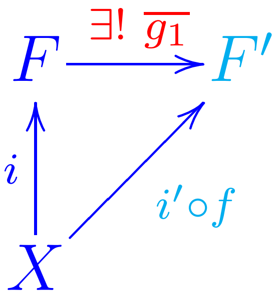


They are the same as

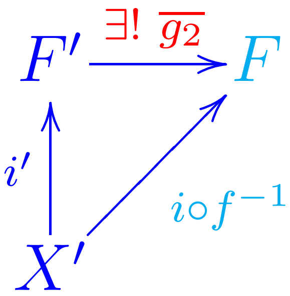
Theorem (7.8)

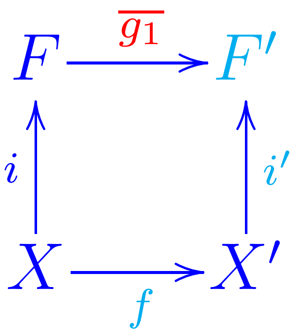
Let F and F' be objects in a concrete category $\mathcal{C}$. Suppose that F is free on the set X and F' is free on the set X'. If $|X| = |X'|$, then F and F' are equivalent.

Proof. Since $|X| = |X'|$, there is a bijection $f : X \to X'$ and hence $f^{-1} : X' \to X$ exists. Let $i : X \to F$ and $i' : X' \to F'$ be the corresponding maps in the definition, i.e.,



and

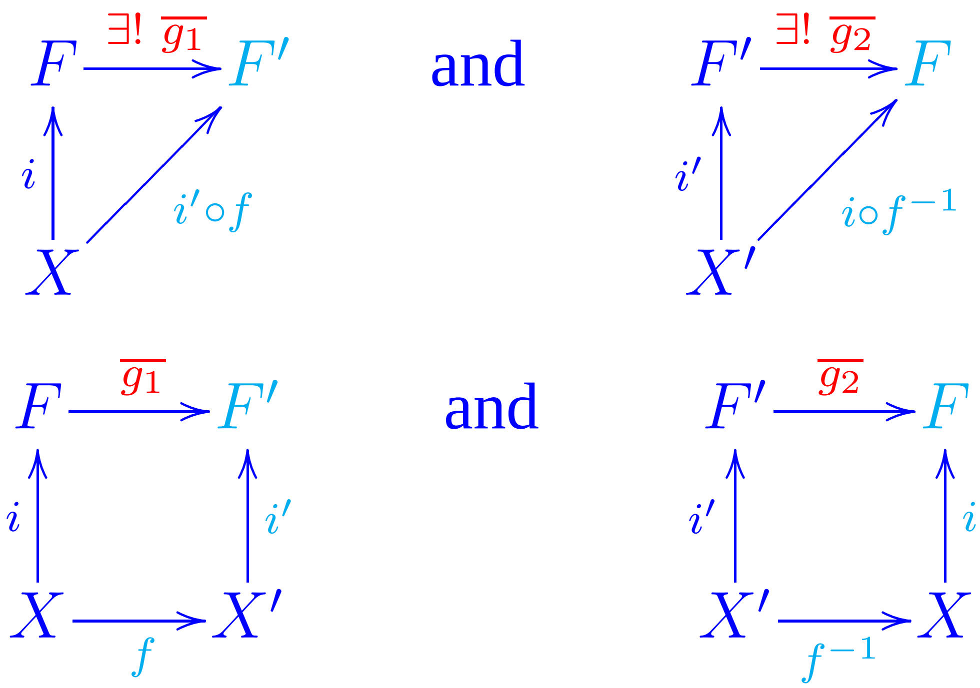




Theorem (7.8)

Let F and F' be objects in a concrete category $\mathcal{C}$. Suppose that F is free on the set X and F' is free on the set X'. If $|X| = |X'|$, then F and F' are equivalent.

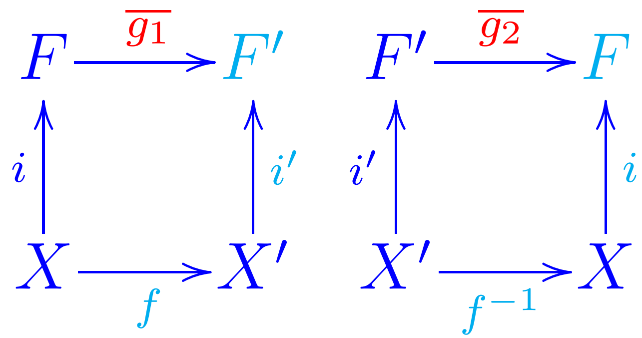
Proof. Since $|X| = |X'|$, there is a bijection $f : X \to X'$ and hence $f^{-1} : X' \to X$ exists. Let $i : X \to F$ and $i' : X' \to F'$ be the corresponding maps in the definition, i.e.,



Theorem (7.8)

Let F and F' be objects in a concrete category $\mathcal{C}$. Suppose that F is free on the set X and F' is free on the set X'. If $|X| = |X'|$, then F and F' are equivalent.

Proof. We already have these two commutative diagrams



Theorem (7.8)

Let F and F' be objects in a concrete category $\mathcal{C}$. Suppose that F is free on the set X and F' is free on the set X'. If $|X| = |X'|$, then F and F' are equivalent.

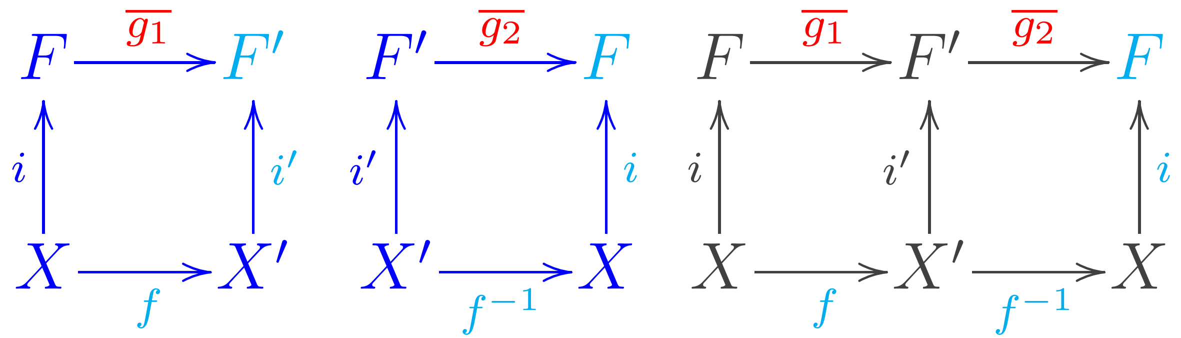
Proof. Combine these two, then we get

$$
\begin{array}{ccc}
F & \xrightarrow{\overline{g_1}} & F' \\
\uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i'} \\
X & \xrightarrow[f]{} & X'
\end{array}
\qquad
\begin{array}{ccc}
F' & \xrightarrow{\overline{g_2}} & F \\
\uparrow{\scriptstyle i'} & & \uparrow{\scriptstyle i} \\
X' & \xrightarrow[f^{-1}]{} & X
\end{array}
$$

Theorem (7.8)

Let F and F' be objects in a concrete category $\mathcal{C}$. Suppose that F is free on the set X and F' is free on the set X'. If $|X| = |X'|$, then F and F' are equivalent.

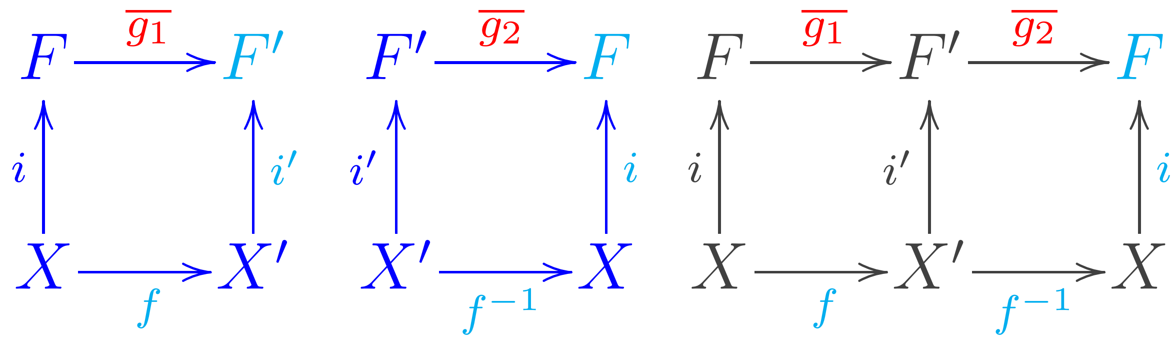
Proof. Combine these two, then we get



Theorem (7.8)

Let F and F' be objects in a concrete category $\mathcal{C}$. Suppose that F is free on the set X and F' is free on the set X'. If $|X| = |X'|$, then F and F' are equivalent.

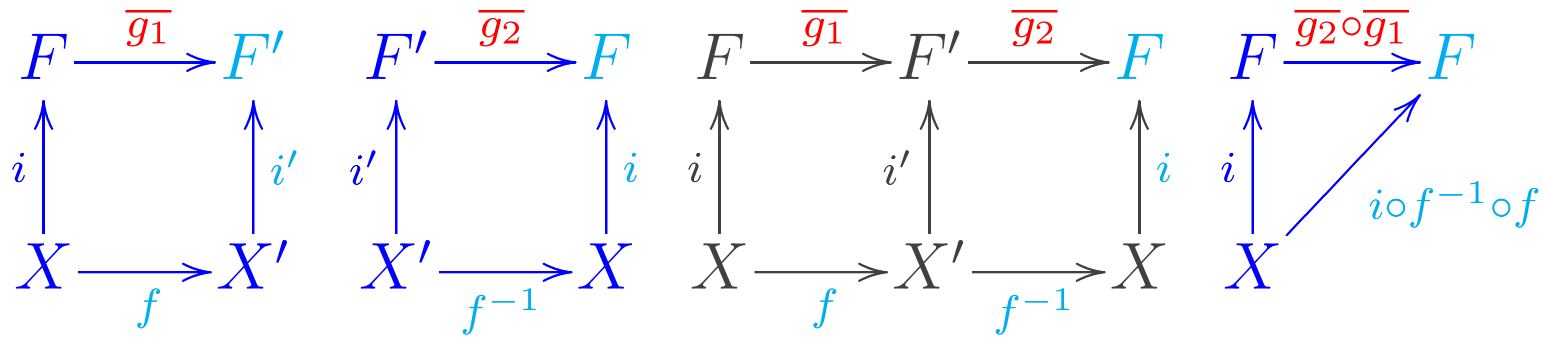
Proof. This is the same as the diagram



Theorem (7.8)

Let F and F' be objects in a concrete category $\mathcal{C}$. Suppose that F is free on the set X and F' is free on the set X'. If $|X| = |X'|$, then F and F' are equivalent.

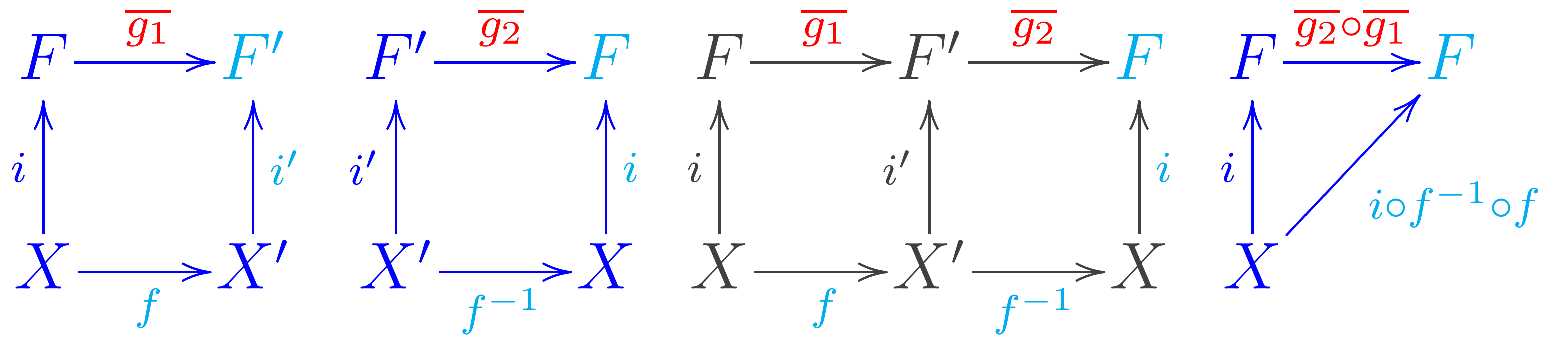
Proof. This is the same as the diagram



Theorem (7.8)

Let F and F' be objects in a concrete category $\mathcal{C}$. Suppose that F is free on the set X and F' is free on the set X'. If $|X| = |X'|$, then F and F' are equivalent.

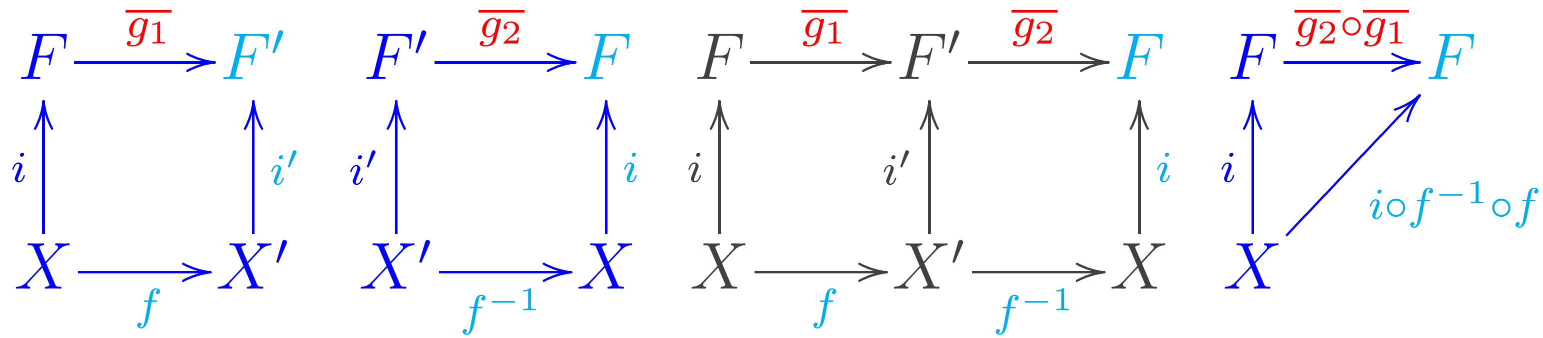
Proof. Note that $i \circ f^{-1} \circ f = i$,



Theorem (7.8)

Let F and F' be objects in a concrete category $\mathcal{C}$. Suppose that F is free on the set X and F' is free on the set X'. If $|X| = |X'|$, then F and F' are equivalent.

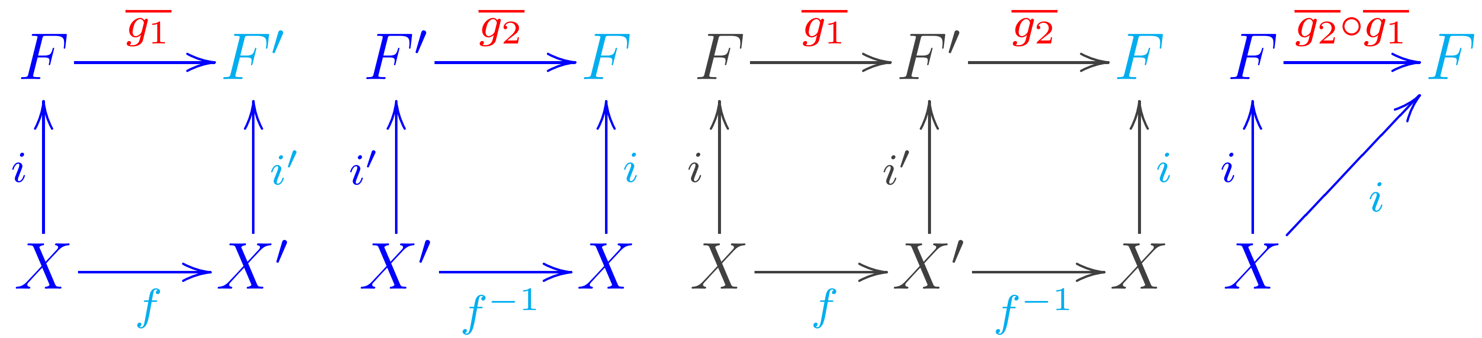
Proof. Note that $i \circ f^{-1} \circ f = i$, so we have



Theorem (7.8)

Let F and F' be objects in a concrete category $\mathcal{C}$. Suppose that F is free on the set X and F' is free on the set X'. If $|X| = |X'|$, then F and F' are equivalent.

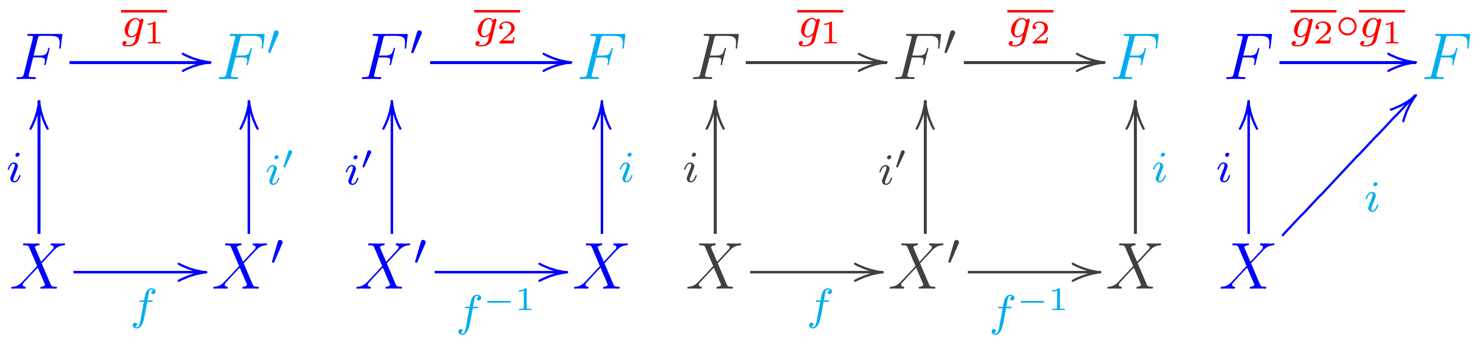
Proof. Note that $i \circ f^{-1} \circ f = i$, so we have



Theorem (7.8)

Let F and F' be objects in a concrete category $\mathcal{C}$. Suppose that F is free on the set X and F' is free on the set X'. If $|X| = |X'|$, then F and F' are equivalent.

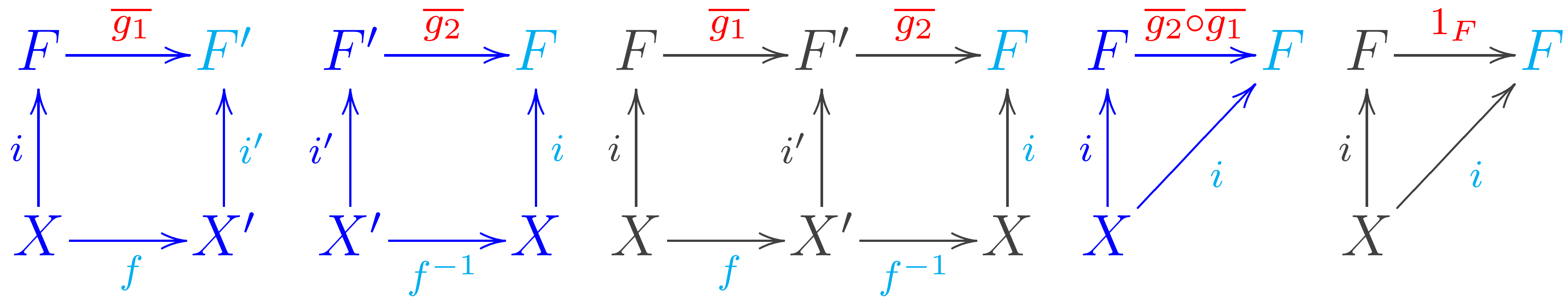
Proof. However, we also have the commutative diagram



Theorem (7.8)

Let F and F' be objects in a concrete category $\mathcal{C}$. Suppose that F is free on the set X and F' is free on the set X'. If $|X| = |X'|$, then F and F' are equivalent.

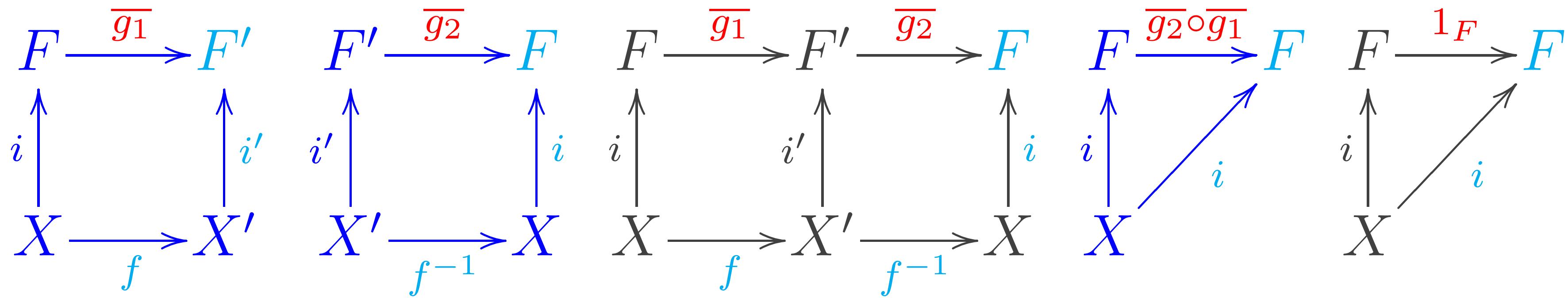
Proof. However, we also have the commutative diagram



Theorem (7.8)

Let F and F' be objects in a concrete category $\mathcal{C}$. Suppose that F is free on the set X and F' is free on the set X'. If $|X| = |X'|$, then F and F' are equivalent.

Proof.

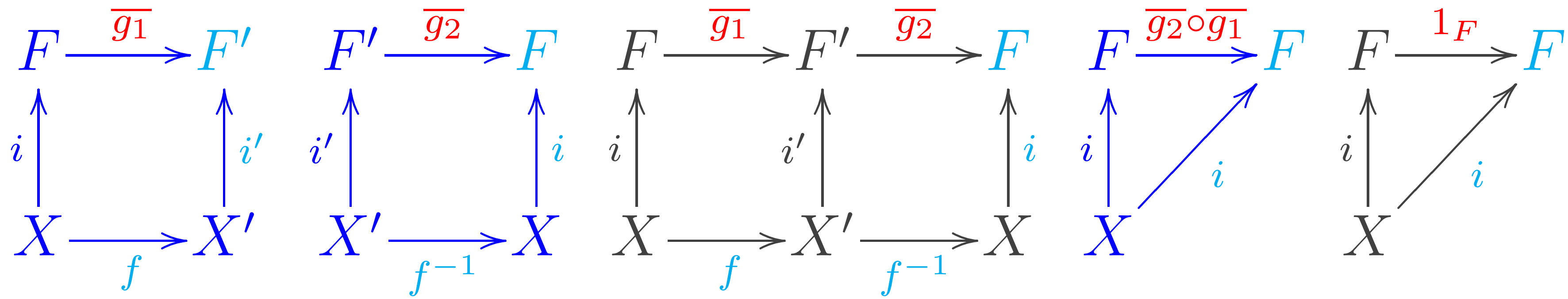


By the uniqueness of the morphism $F \to F$ which makes the diagram work,

Theorem (7.8)

Let F and F' be objects in a concrete category $\mathcal{C}$. Suppose that F is free on the set X and F' is free on the set X'. If $|X| = |X'|$, then F and F' are equivalent.

Proof.

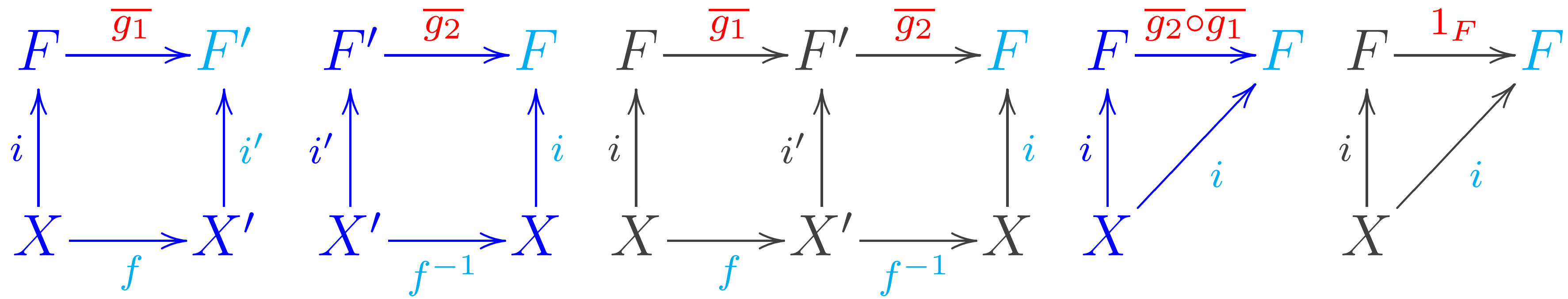


By the uniqueness of the morphism $F \to F$ which makes the diagram work, we have $\overline{g_2} \circ \overline{g_1} = 1_F$.

Theorem (7.8)

Let F and F' be objects in a concrete category $\mathcal{C}$. Suppose that F is free on the set X and F' is free on the set X'. If $|X| = |X'|$, then F and F' are equivalent.

Proof.

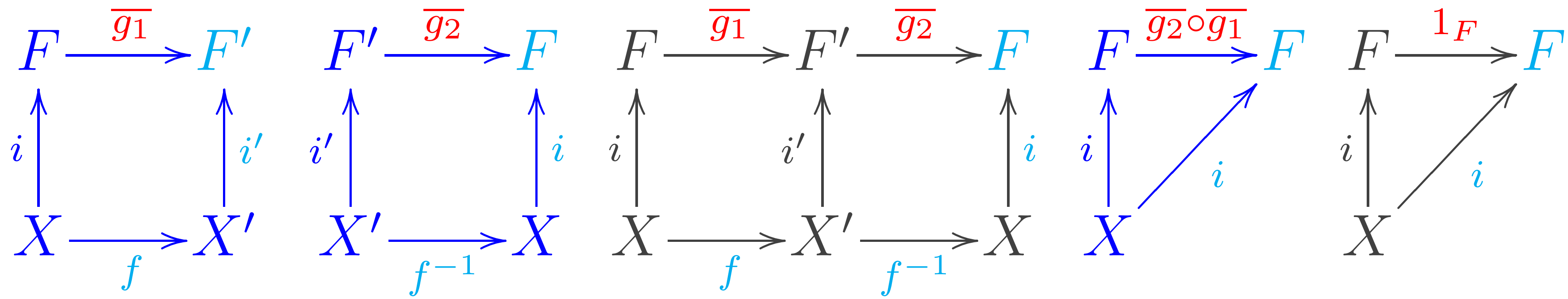


By the uniqueness of the morphism $F \to F$ which makes the diagram work, we have $\overline{g_2} \circ \overline{g_1} = 1_F$. With a similar discussion,

Theorem (7.8)

Let F and F' be objects in a concrete category $\mathcal{C}$. Suppose that F is free on the set X and F' is free on the set X'. If $|X| = |X'|$, then F and F' are equivalent.

Proof.

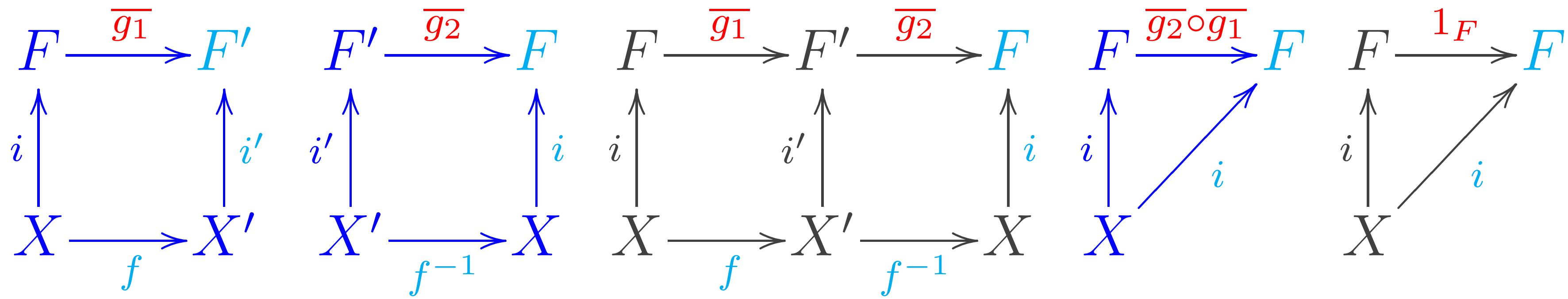


By the uniqueness of the morphism $F \to F$ which makes the diagram work, we have $\overline{g_2} \circ \overline{g_1} = 1_F$. With a similar discussion, we can also show that $\overline{g_1} \circ \overline{g_2} = 1_{F'}$.

Theorem (7.8)

Let F and F' be objects in a concrete category $\mathcal{C}$. Suppose that F is free on the set X and F' is free on the set X'. If $|X| = |X'|$, then F and F' are equivalent.

Proof.



By the uniqueness of the morphism $F \to F$ which makes the diagram work, we have $\overline{g_2} \circ \overline{g_1} = 1_F$. With a similar discussion, we can also show that $\overline{g_1} \circ \overline{g_2} = 1_{F'}$. Hence, F and F' are equivalent.

Universal Mapping Properties

Universal Mapping Properties

Remark. Products, coproducts, and free objects are all defined via universal mapping properties,

Universal Mapping Properties

Remark. Products, coproducts, and free objects are all defined via universal mapping properties, that is, in terms of the existence of certain uniquely determined morphisms.

Universal Mapping Properties

Remark. Products, coproducts, and free objects are all defined via universal mapping properties, that is, in terms of the existence of certain uniquely determined morphisms. We have also seen that any two products

Universal Mapping Properties

Remark. Products, coproducts, and free objects are all defined via universal mapping properties, that is, in terms of the existence of certain uniquely determined morphisms. We have also seen that any two products (or coproducts)

Universal Mapping Properties

Remark. Products, coproducts, and free objects are all defined via universal mapping properties, that is, in terms of the existence of certain uniquely determined morphisms. We have also seen that any two products (or coproducts) for a given family of objects are actually equivalent

Universal Mapping Properties

Remark. Products, coproducts, and free objects are all defined via universal mapping properties, that is, in terms of the existence of certain uniquely determined morphisms. We have also seen that any two products (or coproducts) for a given family of objects are actually equivalent (Theorem (7.3) and (7.5)).

Universal Mapping Properties

Remark. Products, coproducts, and free objects are all defined via universal mapping properties, that is, in terms of the existence of certain uniquely determined morphisms. We have also seen that any two products (or coproducts) for a given family of objects are actually equivalent (Theorem (7.3) and (7.5)). Similarly, any two free objects on the same set are equivalent

Universal Mapping Properties

Remark. Products, coproducts, and free objects are all defined via universal mapping properties, that is, in terms of the existence of certain uniquely determined morphisms. We have also seen that any two products (or coproducts) for a given family of objects are actually equivalent (Theorem (7.3) and (7.5)). Similarly, any two free objects on the same set are equivalent (Theorem (7.8)).

Universal Mapping Properties

Remark. Products, coproducts, and free objects are all defined via universal mapping properties, that is, in terms of the existence of certain uniquely determined morphisms. We have also seen that any two products (or coproducts) for a given family of objects are actually equivalent (Theorem (7.3) and (7.5)). Similarly, any two free objects on the same set are equivalent (Theorem (7.8)). Furthermore, their proofs are all very similar.

Universal Mapping Properties

Remark. Products, coproducts, and free objects are all defined via universal mapping properties, that is, in terms of the existence of certain uniquely determined morphisms. We have also seen that any two products (or coproducts) for a given family of objects are actually equivalent (Theorem (7.3) and (7.5)). Similarly, any two free objects on the same set are equivalent (Theorem (7.8)). Furthermore, their proofs are all very similar. In fact, they are special cases of a single concept.

Universal Mapping Properties

Remark. Products, coproducts, and free objects are all defined via universal mapping properties, that is, in terms of the existence of certain uniquely determined morphisms. We have also seen that any two products (or coproducts) for a given family of objects are actually equivalent (Theorem (7.3) and (7.5)). Similarly, any two free objects on the same set are equivalent (Theorem (7.8)). Furthermore, their proofs are all very similar. In fact, they are special cases of a single concept.

Definition (7.9)**.** Let $\mathcal{C}$ be a category.

Universal Mapping Properties

Remark. Products, coproducts, and free objects are all defined via universal mapping properties, that is, in terms of the existence of certain uniquely determined morphisms. We have also seen that any two products (or coproducts) for a given family of objects are actually equivalent (Theorem (7.3) and (7.5)). Similarly, any two free objects on the same set are equivalent (Theorem (7.8)). Furthermore, their proofs are all very similar. In fact, they are special cases of a single concept.

Definition (7.9)**.** Let $\mathcal{C}$ be a category.

- An object I in $\mathcal{C}$ is said to be universal

Universal Mapping Properties

Remark. Products, coproducts, and free objects are all defined via universal mapping properties, that is, in terms of the existence of certain uniquely determined morphisms. We have also seen that any two products (or coproducts) for a given family of objects are actually equivalent (Theorem (7.3) and (7.5)). Similarly, any two free objects on the same set are equivalent (Theorem (7.8)). Furthermore, their proofs are all very similar. In fact, they are special cases of a single concept.

Definition (7.9)**.** Let $\mathcal{C}$ be a category.

- An object I in $\mathcal{C}$ is said to be universal (or initial)

Universal Mapping Properties

Remark. Products, coproducts, and free objects are all defined via universal mapping properties, that is, in terms of the existence of certain uniquely determined morphisms. We have also seen that any two products (or coproducts) for a given family of objects are actually equivalent (Theorem (7.3) and (7.5)). Similarly, any two free objects on the same set are equivalent (Theorem (7.8)). Furthermore, their proofs are all very similar. In fact, they are special cases of a single concept.

Definition (7.9)**.** Let $\mathcal{C}$ be a category.

- An object I in $\mathcal{C}$ is said to be universal (or initial) if for each object C of $\mathcal{C}$

Universal Mapping Properties

Remark. Products, coproducts, and free objects are all defined via universal mapping properties, that is, in terms of the existence of certain uniquely determined morphisms. We have also seen that any two products (or coproducts) for a given family of objects are actually equivalent (Theorem (7.3) and (7.5)). Similarly, any two free objects on the same set are equivalent (Theorem (7.8)). Furthermore, their proofs are all very similar. In fact, they are special cases of a single concept.

Definition (7.9)**.** Let $\mathcal{C}$ be a category.

- An object I in $\mathcal{C}$ is said to be universal (or initial) if for each object C of $\mathcal{C}$ there exists one and only one morphism $I \to C$.

Universal Mapping Properties

Remark. Products, coproducts, and free objects are all defined via universal mapping properties, that is, in terms of the existence of certain uniquely determined morphisms. We have also seen that any two products (or coproducts) for a given family of objects are actually equivalent (Theorem (7.3) and (7.5)). Similarly, any two free objects on the same set are equivalent (Theorem (7.8)). Furthermore, their proofs are all very similar. In fact, they are special cases of a single concept.

Definition (7.9)**.** Let $\mathcal{C}$ be a category.

- An object I in $\mathcal{C}$ is said to be universal (or initial) if for each object C of $\mathcal{C}$ there exists one and only one morphism $I \to C$.
- An object T in $\mathcal{C}$ is said to be couniversal

Universal Mapping Properties

Remark. Products, coproducts, and free objects are all defined via universal mapping properties, that is, in terms of the existence of certain uniquely determined morphisms. We have also seen that any two products (or coproducts) for a given family of objects are actually equivalent (Theorem (7.3) and (7.5)). Similarly, any two free objects on the same set are equivalent (Theorem (7.8)). Furthermore, their proofs are all very similar. In fact, they are special cases of a single concept.

Definition (7.9). Let $\mathcal{C}$ be a category.

- An object I in $\mathcal{C}$ is said to be universal (or initial) if for each object C of $\mathcal{C}$ there exists one and only one morphism $I \to C$.
- An object T in $\mathcal{C}$ is said to be couniversal (or terminal)

Universal Mapping Properties

Remark. Products, coproducts, and free objects are all defined via universal mapping properties, that is, in terms of the existence of certain uniquely determined morphisms. We have also seen that any two products (or coproducts) for a given family of objects are actually equivalent (Theorem (7.3) and (7.5)). Similarly, any two free objects on the same set are equivalent (Theorem (7.8)). Furthermore, their proofs are all very similar. In fact, they are special cases of a single concept.

Definition (7.9). Let $\mathcal{C}$ be a category.

- An object I in $\mathcal{C}$ is said to be universal (or initial) if for each object C of $\mathcal{C}$ there exists one and only one morphism $I \to C$.
- An object T in $\mathcal{C}$ is said to be couniversal (or terminal) if for each object C of $\mathcal{C}$

Universal Mapping Properties

Remark. Products, coproducts, and free objects are all defined via universal mapping properties, that is, in terms of the existence of certain uniquely determined morphisms. We have also seen that any two products (or coproducts) for a given family of objects are actually equivalent (Theorem (7.3) and (7.5)). Similarly, any two free objects on the same set are equivalent (Theorem (7.8)). Furthermore, their proofs are all very similar. In fact, they are special cases of a single concept.

Definition (7.9)**.** Let $\mathcal{C}$ be a category.

- An object I in $\mathcal{C}$ is said to be universal (or initial) if for each object C of $\mathcal{C}$ there exists one and only one morphism $I \to C$.
- An object T in $\mathcal{C}$ is said to be couniversal (or terminal) if for each object C of $\mathcal{C}$ there exists one and only one morphism $C \to T$.

Universal Mapping Properties

Definition (7.9)**.** Let $\mathcal{C}$ be a category.

- An object I in $\mathcal{C}$ is said to be universal (or initial) if for each object C of $\mathcal{C}$ there exists one and only one morphism $I \to C$.
- An object T in $\mathcal{C}$ is said to be couniversal (or terminal) if for each object C of $\mathcal{C}$ there exists one and only one morphism $C \to T$.

Remark. In your book, you can see that

Universal Mapping Properties

Definition (7.9)**.** Let $\mathcal{C}$ be a category.

- An object I in $\mathcal{C}$ is said to be universal (or initial) if for each object C of $\mathcal{C}$ there exists one and only one morphism $I \to C$.
- An object T in $\mathcal{C}$ is said to be couniversal (or terminal) if for each object C of $\mathcal{C}$ there exists one and only one morphism $C \to T$.

Remark. In your book, you can see that products, coproducts, and free objects

Universal Mapping Properties

Definition (7.9)**.** Let $\mathcal{C}$ be a category.

- An object I in $\mathcal{C}$ is said to be universal (or initial) if for each object C of $\mathcal{C}$ there exists one and only one morphism $I \to C$.
- An object T in $\mathcal{C}$ is said to be couniversal (or terminal) if for each object C of $\mathcal{C}$ there exists one and only one morphism $C \to T$.

Remark. In your book, you can see that products, coproducts, and free objects may be considered as (co)universal objects in suitably chosen categories.

Universal Mapping Properties

Definition (7.9)**.** Let $\mathcal{C}$ be a category.

- An object I in $\mathcal{C}$ is said to be universal (or initial) if for each object C of $\mathcal{C}$ there exists one and only one morphism $I \to C$.
- An object T in $\mathcal{C}$ is said to be couniversal (or terminal) if for each object C of $\mathcal{C}$ there exists one and only one morphism $C \to T$.

Remark. In your book, you can see that products, coproducts, and free objects may be considered as (co)universal objects in suitably chosen categories. We will skip this part to save some time.

Universal Mapping Properties

Definition (7.9)**.** Let $\mathcal{C}$ be a category.

- An object I in $\mathcal{C}$ is said to be universal (or initial) if for each object C of $\mathcal{C}$ there exists one and only one morphism $I \to C$.
- An object T in $\mathcal{C}$ is said to be couniversal (or terminal) if for each object C of $\mathcal{C}$ there exists one and only one morphism $C \to T$.

Example. In the category $\mathcal{G}$ of groups,

Universal Mapping Properties

Definition (7.9)**.** Let $\mathcal{C}$ be a category.

- An object I in $\mathcal{C}$ is said to be universal (or initial) if for each object C of $\mathcal{C}$ there exists one and only one morphism $I \to C$.
- An object T in $\mathcal{C}$ is said to be couniversal (or terminal) if for each object C of $\mathcal{C}$ there exists one and only one morphism $C \to T$.

Example. In the category $\mathcal{G}$ of groups, the trivial group $\{e\}$ is both universal and couniversal.

Universal Mapping Properties

Definition (7.9)**.** Let $\mathcal{C}$ be a category.

- An object I in $\mathcal{C}$ is said to be universal (or initial) if for each object C of $\mathcal{C}$ there exists one and only one morphism $I \to C$.
- An object T in $\mathcal{C}$ is said to be couniversal (or terminal) if for each object C of $\mathcal{C}$ there exists one and only one morphism $C \to T$.

Example. In the category $\mathcal{G}$ of groups, the trivial group $\{e\}$ is both universal and couniversal.

Proof.

Universal Mapping Properties

Definition (7.9)**.** Let $\mathcal{C}$ be a category.

- An object I in $\mathcal{C}$ is said to be universal (or initial) if for each object C of $\mathcal{C}$ there exists one and only one morphism $I \to C$.
- An object T in $\mathcal{C}$ is said to be couniversal (or terminal) if for each object C of $\mathcal{C}$ there exists one and only one morphism $C \to T$.

Example. In the category $\mathcal{G}$ of groups, the trivial group $\{e\}$ is both universal and couniversal.

Proof. It is clear that for any group G,

Universal Mapping Properties

Definition (7.9)**.** Let $\mathcal{C}$ be a category.

- An object I in $\mathcal{C}$ is said to be universal (or initial) if for each object C of $\mathcal{C}$ there exists one and only one morphism $I \to C$.
- An object T in $\mathcal{C}$ is said to be couniversal (or terminal) if for each object C of $\mathcal{C}$ there exists one and only one morphism $C \to T$.

Example. In the category $\mathcal{G}$ of groups, the trivial group $\{e\}$ is both universal and couniversal.

Proof. It is clear that for any group G, there is one and only one group homomorphism $\{e\} \to G$,

Universal Mapping Properties

Definition (7.9)**.** Let $\mathcal{C}$ be a category.

- An object I in $\mathcal{C}$ is said to be universal (or initial) if for each object C of $\mathcal{C}$ there exists one and only one morphism $I \to C$.
- An object T in $\mathcal{C}$ is said to be couniversal (or terminal) if for each object C of $\mathcal{C}$ there exists one and only one morphism $C \to T$.

Example. In the category $\mathcal{G}$ of groups, the trivial group $\{e\}$ is both universal and couniversal.

Proof. It is clear that for any group G, there is one and only one group homomorphism $\{e\} \to G$, namely the map defined by $e \mapsto e$.

Universal Mapping Properties

Definition (7.9)**.** Let $\mathcal{C}$ be a category.

- An object I in $\mathcal{C}$ is said to be universal (or initial) if for each object C of $\mathcal{C}$ there exists one and only one morphism $I \to C$.
- An object T in $\mathcal{C}$ is said to be couniversal (or terminal) if for each object C of $\mathcal{C}$ there exists one and only one morphism $C \to T$.

Example. In the category $\mathcal{G}$ of groups, the trivial group $\{e\}$ is both universal and couniversal.

Proof. It is clear that for any group G, there is one and only one group homomorphism $\{e\} \to G$, namely the map defined by $e \mapsto e$. Hence $\{e\}$ is universal.

Universal Mapping Properties

Definition (7.9)**.** Let $\mathcal{C}$ be a category.

- An object I in $\mathcal{C}$ is said to be universal (or initial) if for each object C of $\mathcal{C}$ there exists one and only one morphism $I \to C$.
- An object T in $\mathcal{C}$ is said to be couniversal (or terminal) if for each object C of $\mathcal{C}$ there exists one and only one morphism $C \to T$.

Example. In the category $\mathcal{G}$ of groups, the trivial group $\{e\}$ is both universal and couniversal.

Proof. It is clear that for any group G, there is one and only one group homomorphism $\{e\} \to G$, namely the map defined by $e \mapsto e$. Hence $\{e\}$ is universal. It is also clear that for any group G,

Universal Mapping Properties

Definition (7.9)**.** Let $\mathcal{C}$ be a category.

- An object I in $\mathcal{C}$ is said to be universal (or initial) if for each object C of $\mathcal{C}$ there exists one and only one morphism $I \to C$.
- An object T in $\mathcal{C}$ is said to be couniversal (or terminal) if for each object C of $\mathcal{C}$ there exists one and only one morphism $C \to T$.

Example. In the category $\mathcal{G}$ of groups, the trivial group $\{e\}$ is both universal and couniversal.

Proof. It is clear that for any group G, there is one and only one group homomorphism $\{e\} \to G$, namely the map defined by $e \mapsto e$. Hence $\{e\}$ is universal. It is also clear that for any group G, the only group homomorphism $G \to \{e\}$

Universal Mapping Properties

Definition (7.9)**.** Let $\mathcal{C}$ be a category.

- An object I in $\mathcal{C}$ is said to be universal (or initial) if for each object C of $\mathcal{C}$ there exists one and only one morphism $I \to C$.
- An object T in $\mathcal{C}$ is said to be couniversal (or terminal) if for each object C of $\mathcal{C}$ there exists one and only one morphism $C \to T$.

Example. In the category $\mathcal{G}$ of groups, the trivial group $\{e\}$ is both universal and couniversal.

Proof. It is clear that for any group G, there is one and only one group homomorphism $\{e\} \to G$, namely the map defined by $e \mapsto e$. Hence $\{e\}$ is universal. It is also clear that for any group G, the only group homomorphism $G \to \{e\}$ is the map defined by $a \mapsto e$ for all $a \in G$.

Universal Mapping Properties

Definition (7.9)**.** Let $\mathcal{C}$ be a category.

- An object I in $\mathcal{C}$ is said to be universal (or initial) if for each object C of $\mathcal{C}$ there exists one and only one morphism $I \to C$.
- An object T in $\mathcal{C}$ is said to be couniversal (or terminal) if for each object C of $\mathcal{C}$ there exists one and only one morphism $C \to T$.

Example. In the category $\mathcal{G}$ of groups, the trivial group $\{e\}$ is both universal and couniversal.

Proof. It is clear that for any group G, there is one and only one group homomorphism $\{e\} \to G$, namely the map defined by $e \mapsto e$. Hence $\{e\}$ is universal. It is also clear that for any group G, the only group homomorphism $G \to \{e\}$ is the map defined by $a \mapsto e$ for all $a \in G$. Hence, $\{e\}$ is couniversal.

Uniqueness of universal objects

Theorem (7.10)**.** Any two universal objects in a category $\mathcal{C}$ are equivalent.

Uniqueness of universal objects

Theorem (7.10)**.** Any two universal [resp. couniversal] objects in a category $\mathcal{C}$ are equivalent.

Uniqueness of universal objects

Theorem (7.10)**.** Any two universal [resp. couniversal] objects in a category $\mathcal{C}$ are equivalent.

Proof.

Uniqueness of universal objects

Theorem (7.10)**.** Any two universal [resp. couniversal] objects in a category $\mathcal{C}$ are equivalent.

Proof. Let I and J be universal objects in $\mathcal{C}$.

Uniqueness of universal objects

Theorem (7.10)**.** Any two universal [resp. couniversal] objects in a category $\mathcal{C}$ are equivalent.

Proof. Let I and J be universal objects in $\mathcal{C}$. Since I is universal,

Uniqueness of universal objects

Theorem (7.10)**.** Any two universal [resp. couniversal] objects in a category $\mathcal{C}$ are equivalent.

Proof. Let I and J be universal objects in $\mathcal{C}$. Since I is universal, there is a unique morphism $f : I \longrightarrow J$.

Uniqueness of universal objects

Theorem (7.10)**.** Any two universal [resp. couniversal] objects in a category $\mathcal{C}$ are equivalent.

Proof. Let I and J be universal objects in $\mathcal{C}$. Since I is universal, there is a unique morphism $f : I \longrightarrow J$. Similarly, since J is universal,

Uniqueness of universal objects

Theorem (7.10)**.** Any two universal [resp. couniversal] objects in a category $\mathcal{C}$ are equivalent.

Proof. Let I and J be universal objects in $\mathcal{C}$. Since I is universal, there is a unique morphism $f : I \to J$. Similarly, since J is universal, there is a unique morphism $g : J \to I$.

Uniqueness of universal objects

Theorem (7.10)**.** Any two universal [resp. couniversal] objects in a category $\mathcal{C}$ are equivalent.

Proof. Let I and J be universal objects in $\mathcal{C}$. Since I is universal, there is a unique morphism $f : I \longrightarrow J$. Similarly, since J is universal, there is a unique morphism $g : J \longrightarrow I$. The composite $g \circ f : I \longrightarrow I$ is a morphism of $\mathcal{C}$.

Uniqueness of universal objects

Theorem (7.10)**.** Any two universal [resp. couniversal] objects in a category $\mathcal{C}$ are equivalent.

Proof. Let I and J be universal objects in $\mathcal{C}$. Since I is universal, there is a unique morphism $f : I \longrightarrow J$. Similarly, since J is universal, there is a unique morphism $g : J \longrightarrow I$. The composite $g \circ f : I \longrightarrow I$ is a morphism of $\mathcal{C}$. But, $1_I : I \longrightarrow I$ is also a morphism of $\mathcal{C}$.

Uniqueness of universal objects

Theorem (7.10)**.** Any two universal [resp. couniversal] objects in a category $\mathcal{C}$ are equivalent.

Proof. Let I and J be universal objects in $\mathcal{C}$. Since I is universal, there is a unique morphism $f : I \longrightarrow J$. Similarly, since J is universal, there is a unique morphism $g : J \longrightarrow I$. The composite $g \circ f : I \longrightarrow I$ is a morphism of $\mathcal{C}$. But, $1_I : I \longrightarrow I$ is also a morphism of $\mathcal{C}$. The universality of I implies that

Uniqueness of universal objects

Theorem (7.10)**.** Any two universal [resp. couniversal] objects in a category $\mathcal{C}$ are equivalent.

Proof. Let I and J be universal objects in $\mathcal{C}$. Since I is universal, there is a unique morphism $f : I \longrightarrow J$. Similarly, since J is universal, there is a unique morphism $g : J \longrightarrow I$. The composite $g \circ f : I \longrightarrow I$ is a morphism of $\mathcal{C}$. But, $1_I : I \longrightarrow I$ is also a morphism of $\mathcal{C}$. The universality of I implies that there is a unique morphism $I \longrightarrow I$,

Uniqueness of universal objects

Theorem (7.10)**.** Any two universal [resp. couniversal] objects in a category $\mathcal{C}$ are equivalent.

Proof. Let I and J be universal objects in $\mathcal{C}$. Since I is universal, there is a unique morphism $f : I \longrightarrow J$. Similarly, since J is universal, there is a unique morphism $g : J \longrightarrow I$. The composite $g \circ f : I \longrightarrow I$ is a morphism of $\mathcal{C}$. But, $1_I : I \longrightarrow I$ is also a morphism of $\mathcal{C}$. The universality of I implies that there is a unique morphism $I \longrightarrow I$, whence $g \circ f = 1_I$.

Uniqueness of universal objects

Theorem (7.10)**.** Any two universal [resp. couniversal] objects in a category $\mathcal{C}$ are equivalent.

Proof. Let I and J be universal objects in $\mathcal{C}$. Since I is universal, there is a unique morphism $f : I \longrightarrow J$. Similarly, since J is universal, there is a unique morphism $g : J \longrightarrow I$. The composite $g \circ f : I \longrightarrow I$ is a morphism of $\mathcal{C}$. But, $1_I : I \longrightarrow I$ is also a morphism of $\mathcal{C}$. The universality of I implies that there is a unique morphism $I \longrightarrow I$, whence $g \circ f = 1_I$. Similarly,

Uniqueness of universal objects

Theorem (7.10)**.** Any two universal [resp. couniversal] objects in a category $\mathcal{C}$ are equivalent.

Proof. Let I and J be universal objects in $\mathcal{C}$. Since I is universal, there is a unique morphism $f : I \to J$. Similarly, since J is universal, there is a unique morphism $g : J \to I$. The composite $g \circ f : I \to I$ is a morphism of $\mathcal{C}$. But, $1_I : I \to I$ is also a morphism of $\mathcal{C}$. The universality of I implies that there is a unique morphism $I \to I$, whence $g \circ f = 1_I$. Similarly, since $f \circ g : J \to J$

Uniqueness of universal objects

Theorem (7.10)**.** Any two universal [resp. couniversal] objects in a category $\mathcal{C}$ are equivalent.

Proof. Let I and J be universal objects in $\mathcal{C}$. Since I is universal, there is a unique morphism $f : I \longrightarrow J$. Similarly, since J is universal, there is a unique morphism $g : J \longrightarrow I$. The composite $g \circ f : I \longrightarrow I$ is a morphism of $\mathcal{C}$. But, $1_I : I \longrightarrow I$ is also a morphism of $\mathcal{C}$. The universality of I implies that there is a unique morphism $I \longrightarrow I$, whence $g \circ f = 1_I$. Similarly, since $f \circ g : J \longrightarrow J$ and $1_J : J \longrightarrow J$

Uniqueness of universal objects

Theorem (7.10)**.** Any two universal [resp. couniversal] objects in a category $\mathcal{C}$ are equivalent.

Proof. Let I and J be universal objects in $\mathcal{C}$. Since I is universal, there is a unique morphism $f : I \longrightarrow J$. Similarly, since J is universal, there is a unique morphism $g : J \longrightarrow I$. The composite $g \circ f : I \longrightarrow I$ is a morphism of $\mathcal{C}$. But, $1_I : I \longrightarrow I$ is also a morphism of $\mathcal{C}$. The universality of I implies that there is a unique morphism $I \longrightarrow I$, whence $g \circ f = 1_I$. Similarly, since $f \circ g : J \longrightarrow J$ and $1_J : J \longrightarrow J$ are both morphisms of $\mathcal{C}$,

Uniqueness of universal objects

Theorem (7.10)**.** Any two universal [resp. couniversal] objects in a category $\mathcal{C}$ are equivalent.

Proof. Let I and J be universal objects in $\mathcal{C}$. Since I is universal, there is a unique morphism $f : I \to J$. Similarly, since J is universal, there is a unique morphism $g : J \to I$. The composite $g \circ f : I \to I$ is a morphism of $\mathcal{C}$. But, $1_I : I \to I$ is also a morphism of $\mathcal{C}$. The universality of I implies that there is a unique morphism $I \to I$, whence $g \circ f = 1_I$. Similarly, since $f \circ g : J \to J$ and $1_J : J \to J$ are both morphisms of $\mathcal{C}$, by the uniqueness of the morphism $J \to J$,

Uniqueness of universal objects

Theorem (7.10)**.** Any two universal [resp. couniversal] objects in a category $\mathcal{C}$ are equivalent.

Proof. Let I and J be universal objects in $\mathcal{C}$. Since I is universal, there is a unique morphism $f : I \longrightarrow J$. Similarly, since J is universal, there is a unique morphism $g : J \longrightarrow I$. The composite $g \circ f : I \longrightarrow I$ is a morphism of $\mathcal{C}$. But, $1_I : I \longrightarrow I$ is also a morphism of $\mathcal{C}$. The universality of I implies that there is a unique morphism $I \longrightarrow I$, whence $g \circ f = 1_I$. Similarly, since $f \circ g : J \longrightarrow J$ and $1_J : J \longrightarrow J$ are both morphisms of $\mathcal{C}$, by the uniqueness of the morphism $J \longrightarrow J$, we have $f \circ g = 1_J$.

Uniqueness of universal objects

Theorem (7.10)**.** Any two universal [resp. couniversal] objects in a category $\mathcal{C}$ are equivalent.

Proof. Let I and J be universal objects in $\mathcal{C}$. Since I is universal, there is a unique morphism $f : I \longrightarrow J$. Similarly, since J is universal, there is a unique morphism $g : J \longrightarrow I$. The composite $g \circ f : I \longrightarrow I$ is a morphism of $\mathcal{C}$. But, $1_I : I \longrightarrow I$ is also a morphism of $\mathcal{C}$. The universality of I implies that there is a unique morphism $I \longrightarrow I$, whence $g \circ f = 1_I$. Similarly, since $f \circ g : J \longrightarrow J$ and $1_J : J \longrightarrow J$ are both morphisms of $\mathcal{C}$, by the uniqueness of the morphism $J \longrightarrow J$, we have $f \circ g = 1_J$. Therefore, I and J are equivalent.

Uniqueness of universal objects

Theorem (7.10)**.** Any two universal [resp. couniversal] objects in a category $\mathcal{C}$ are equivalent.

Proof. Let I and J be universal objects in $\mathcal{C}$. Since I is universal, there is a unique morphism $f : I \longrightarrow J$. Similarly, since J is universal, there is a unique morphism $g : J \longrightarrow I$. The composite $g \circ f : I \longrightarrow I$ is a morphism of $\mathcal{C}$. But, $1_I : I \longrightarrow I$ is also a morphism of $\mathcal{C}$. The universality of I implies that there is a unique morphism $I \longrightarrow I$, whence $g \circ f = 1_I$. Similarly, since $f \circ g : J \longrightarrow J$ and $1_J : J \longrightarrow J$ are both morphisms of $\mathcal{C}$, by the uniqueness of the morphism $J \longrightarrow J$, we have $f \circ g = 1_J$. Therefore, I and J are equivalent.

By "reversing the arrows",

Uniqueness of universal objects

Theorem (7.10)**.** Any two universal [resp. couniversal] objects in a category $\mathcal{C}$ are equivalent.

Proof. Let I and J be universal objects in $\mathcal{C}$. Since I is universal, there is a unique morphism $f : I \longrightarrow J$. Similarly, since J is universal, there is a unique morphism $g : J \longrightarrow I$. The composite $g \circ f : I \longrightarrow I$ is a morphism of $\mathcal{C}$. But, $1_I : I \longrightarrow I$ is also a morphism of $\mathcal{C}$. The universality of I implies that there is a unique morphism $I \longrightarrow I$, whence $g \circ f = 1_I$. Similarly, since $f \circ g : J \longrightarrow J$ and $1_J : J \longrightarrow J$ are both morphisms of $\mathcal{C}$, by the uniqueness of the morphism $J \longrightarrow J$, we have $f \circ g = 1_J$. Therefore, I and J are equivalent.

By "reversing the arrows", we have a proof for couniversal objects being unique.

Exercise for Section I.7

1, 2, 3, 4, 7, 8.

Chapter I

Section I.8: Direct Products and Direct Sums

Direct Products of Groups

Definition/Remark. Let $\{G_i \mid i \in I\}$ be a family of groups.

Direct Products of Groups

Definition/Remark. Let $\{G_i \mid i \in I\}$ be a family of groups.

- Define a binary operation on the Cartesian product $\prod_{i \in I} G_i$ as follows:

Direct Products of Groups

Definition/Remark. Let $\{G_i \mid i \in I\}$ be a family of groups.

- Define a binary operation on the Cartesian product $\prod_{i\in I} G_i$ as follows: for $(a_i)_{i\in I}, (b_i)_{i\in I} \in \prod_{i\in I} G_i$,

Direct Products of Groups

Definition/Remark. Let $\{G_i \mid i \in I\}$ be a family of groups.

- Define a binary operation on the Cartesian product $\prod_{i \in I} G_i$ as follows: for $(a_i)_{i \in I}, (b_i)_{i \in I} \in \prod_{i \in I} G_i$,

$$(a_i)_{i \in I}(b_i)_{i \in I} := (a_i b_i)_{i \in I}.$$

Direct Products of Groups

Definition/Remark. Let $\{G_i \mid i \in I\}$ be a family of groups.

- Define a binary operation on the Cartesian product $\prod_{i\in I} G_i$ as follows: for $(a_i)_{i\in I}, (b_i)_{i\in I} \in \prod_{i\in I} G_i$,

$$(a_i)_{i\in I}(b_i)_{i\in I} := (a_i b_i)_{i\in I}.$$

  Since each G_i is a group,

Direct Products of Groups

Definition/Remark. Let $\{G_i \mid i \in I\}$ be a family of groups.

- Define a binary operation on the Cartesian product $\prod_{i \in I} G_i$ as follows: for $(a_i)_{i \in I}, (b_i)_{i \in I} \in \prod_{i \in I} G_i$,

$$(a_i)_{i \in I}(b_i)_{i \in I} := (a_i b_i)_{i \in I}.$$

Since each G_i is a group, $a_i b_i \in G_i$ for each $i \in I$

Direct Products of Groups

Definition/Remark. Let $\{G_i \mid i \in I\}$ be a family of groups.

- Define a binary operation on the Cartesian product $\prod_{i \in I} G_i$ as follows: for $(a_i)_{i \in I}, (b_i)_{i \in I} \in \prod_{i \in I} G_i$,

$$(a_i)_{i \in I}(b_i)_{i \in I} := (a_i b_i)_{i \in I}.$$

  Since each G_i is a group, $a_i b_i \in G_i$ for each $i \in I$ and thus $(a_i b_i)_{i \in I} \in \prod_{i \in I} G_i$.

Direct Products of Groups

Definition/Remark. Let $\{G_i \mid i \in I\}$ be a family of groups.

- Define a binary operation on the Cartesian product $\prod_{i\in I} G_i$ as follows: for $(a_i)_{i\in I}, (b_i)_{i\in I} \in \prod_{i\in I} G_i$,
$$(a_i)_{i\in I}(b_i)_{i\in I} := (a_ib_i)_{i\in I}.$$
Since each G_i is a group, $a_ib_i \in G_i$ for each $i \in I$ and thus $(a_ib_i)_{i\in I} \in \prod_{i\in I} G_i$. Therefore, this binary operation is well-defined.

Direct Products of Groups

Definition/Remark. Let $\{G_i \mid i \in I\}$ be a family of groups.

- Define a binary operation on the Cartesian product $\prod_{i\in I} G_i$ as follows: for $(a_i)_{i\in I}, (b_i)_{i\in I} \in \prod_{i\in I} G_i$,

$$(a_i)_{i\in I}(b_i)_{i\in I} := (a_ib_i)_{i\in I}.$$

  Since each G_i is a group, $a_ib_i \in G_i$ for each $i \in I$ and thus $(a_ib_i)_{i\in I} \in \prod_{i\in I} G_i$. Therefore, this binary operation is well-defined.

- $\prod_{i\in I} G_i$, together with this binary operation,

Direct Products of Groups

Definition/Remark. Let $\{G_i \mid i \in I\}$ be a family of groups.

- Define a binary operation on the Cartesian product $\prod_{i \in I} G_i$ as follows: for $(a_i)_{i \in I}, (b_i)_{i \in I} \in \prod_{i \in I} G_i$,

$$(a_i)_{i \in I}(b_i)_{i \in I} := (a_i b_i)_{i \in I}.$$

  Since each G_i is a group, $a_i b_i \in G_i$ for each $i \in I$ and thus $(a_i b_i)_{i \in I} \in \prod_{i \in I} G_i$. Therefore, this binary operation is well-defined.

- $\prod_{i \in I} G_i$, together with this binary operation, is called the direct product

Direct Products of Groups

Definition/Remark. Let $\{G_i \mid i \in I\}$ be a family of groups.

- Define a binary operation on the Cartesian product $\prod_{i\in I} G_i$ as follows: for $(a_i)_{i\in I}, (b_i)_{i\in I} \in \prod_{i\in I} G_i$,

$$(a_i)_{i\in I}(b_i)_{i\in I} := (a_ib_i)_{i\in I}.$$

  Since each G_i is a group, $a_ib_i \in G_i$ for each $i \in I$ and thus $(a_ib_i)_{i\in I} \in \prod_{i\in I} G_i$. Therefore, this binary operation is well-defined.

- $\prod_{i\in I} G_i$, together with this binary operation, is called the direct product (or complete direct sum) of $\{G_i \mid i \in I\}$.

Direct Products of Groups

Definition/Remark. Let $\{G_i \mid i \in I\}$ be a family of groups.

- Define a binary operation on the Cartesian product $\prod_{i \in I} G_i$ as follows: for $(a_i)_{i \in I}, (b_i)_{i \in I} \in \prod_{i \in I} G_i$,

$$(a_i)_{i \in I}(b_i)_{i \in I} := (a_i b_i)_{i \in I}.$$

  Since each G_i is a group, $a_i b_i \in G_i$ for each $i \in I$ and thus $(a_i b_i)_{i \in I} \in \prod_{i \in I} G_i$. Therefore, this binary operation is well-defined.

- $\prod_{i \in I} G_i$, together with this binary operation, is called the direct product (or complete direct sum) of $\{G_i \mid i \in I\}$.

Theorem (8.1)**.** Let $\{G_i \mid i \in I\}$ be a family of groups.

Direct Products of Groups

Definition/Remark. Let $\{G_i \mid i \in I\}$ be a family of groups.

- Define a binary operation on the Cartesian product $\prod_{i\in I} G_i$ as follows: for $(a_i)_{i\in I}, (b_i)_{i\in I} \in \prod_{i\in I} G_i$,

$$(a_i)_{i\in I}(b_i)_{i\in I} := (a_ib_i)_{i\in I}.$$

  Since each G_i is a group, $a_ib_i \in G_i$ for each $i \in I$ and thus $(a_ib_i)_{i\in I} \in \prod_{i\in I} G_i$. Therefore, this binary operation is well-defined.

- $\prod_{i\in I} G_i$, together with this binary operation, is called the direct product (or complete direct sum) of $\{G_i \mid i \in I\}$.

Theorem (8.1)**.** Let $\{G_i \mid i \in I\}$ be a family of groups.

- The direct product $\prod_{i\in I} G_i$ is a group.

Direct Products of Groups

Definition/Remark. Let $\{G_i \mid i \in I\}$ be a family of groups.

- Define a binary operation on the Cartesian product $\prod_{i \in I} G_i$ as follows: for $(a_i)_{i \in I}, (b_i)_{i \in I} \in \prod_{i \in I} G_i$,
$$(a_i)_{i \in I}(b_i)_{i \in I} := (a_i b_i)_{i \in I}.$$
Since each G_i is a group, $a_i b_i \in G_i$ for each $i \in I$ and thus $(a_i b_i)_{i \in I} \in \prod_{i \in I} G_i$. Therefore, this binary operation is well-defined.
- $\prod_{i \in I} G_i$, together with this binary operation, is called the direct product (or complete direct sum) of $\{G_i \mid i \in I\}$.

Theorem (8.1). Let $\{G_i \mid i \in I\}$ be a family of groups.

- The direct product $\prod_{i \in I} G_i$ is a group.
- For each $k \in I$,

Direct Products of Groups

Definition/Remark. Let $\{G_i \mid i \in I\}$ be a family of groups.

- Define a binary operation on the Cartesian product $\prod_{i\in I} G_i$ as follows: for $(a_i)_{i\in I}, (b_i)_{i\in I} \in \prod_{i\in I} G_i$,
$$(a_i)_{i\in I}(b_i)_{i\in I} := (a_i b_i)_{i\in I}.$$
Since each G_i is a group, $a_i b_i \in G_i$ for each $i \in I$ and thus $(a_i b_i)_{i\in I} \in \prod_{i\in I} G_i$. Therefore, this binary operation is well-defined.
- $\prod_{i\in I} G_i$, together with this binary operation, is called the direct product (or complete direct sum) of $\{G_i \mid i \in I\}$.

Theorem (8.1)**.** Let $\{G_i \mid i \in I\}$ be a family of groups.

- The direct product $\prod_{i\in I} G_i$ is a group.
- For each $k \in I$, the canonical projection $\pi_k : \prod_{i\in I} G_i \to G_k$

Direct Products of Groups

Definition/Remark. Let $\{G_i \mid i \in I\}$ be a family of groups.

- Define a binary operation on the Cartesian product $\prod_{i \in I} G_i$ as follows: for $(a_i)_{i \in I}, (b_i)_{i \in I} \in \prod_{i \in I} G_i$,

$$(a_i)_{i \in I}(b_i)_{i \in I} := (a_i b_i)_{i \in I}.$$

  Since each G_i is a group, $a_i b_i \in G_i$ for each $i \in I$ and thus $(a_i b_i)_{i \in I} \in \prod_{i \in I} G_i$. Therefore, this binary operation is well-defined.

- $\prod_{i \in I} G_i$, together with this binary operation, is called the direct product (or complete direct sum) of $\{G_i \mid i \in I\}$.

Theorem (8.1)**.** Let $\{G_i \mid i \in I\}$ be a family of groups.

- The direct product $\prod_{i \in I} G_i$ is a group.
- For each $k \in I$, the canonical projection $\pi_k : \prod_{i \in I} G_i \to G_k$ is an epimorphism of groups.

Theorem (8.1)

Let $\{G_i \mid i \in I\}$ be a family of groups.

Theorem (8.1)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- The direct product $\prod_{i \in I} G_i$ is a group.

Theorem (8.1)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- The direct product $\prod_{i \in I} G_i$ is a group.

Proof.

Theorem (8.1)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- The direct product $\prod_{i \in I} G_i$ is a group.

Proof. The binary operation is associative because

Theorem (8.1)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- The direct product $\prod_{i \in I} G_i$ is a group.

Proof. The binary operation is associative because

$$\big((a_i)_{i \in I}(b_i)_{i \in I}\big)(c_i)_{i \in I}$$

Theorem (8.1)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- The direct product $\prod_{i\in I} G_i$ is a group.

Proof. The binary operation is associative because

$$\big((a_i)_{i\in I}(b_i)_{i\in I}\big)(c_i)_{i\in I} = (a_ib_i)_{i\in I}(c_i)_{i\in I}$$

Theorem (8.1)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- The direct product $\prod_{i \in I} G_i$ is a group.

Proof. The binary operation is associative because

$$\big((a_i)_{i\in I}(b_i)_{i\in I}\big)(c_i)_{i\in I} = (a_ib_i)_{i\in I}(c_i)_{i\in I} = \big((a_ib_i)c_i\big)_{i\in I}$$

Theorem (8.1)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- The direct product $\prod_{i\in I} G_i$ is a group.

Proof. The binary operation is associative because

$$\big((a_i)_{i\in I}(b_i)_{i\in I}\big)(c_i)_{i\in I} = (a_ib_i)_{i\in I}(c_i)_{i\in I} = \big((a_ib_i)c_i\big)_{i\in I}$$
$$= \big(a_i(b_ic_i)\big)_{i\in I}$$

Theorem (8.1)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- The direct product $\prod_{i \in I} G_i$ is a group.

Proof. The binary operation is associative because

$$\big((a_i)_{i\in I}(b_i)_{i\in I}\big)(c_i)_{i\in I} = (a_ib_i)_{i\in I}(c_i)_{i\in I} = \big((a_ib_i)c_i\big)_{i\in I}$$

$$= \big(a_i(b_ic_i)\big)_{i\in I} = (a_i)_{i\in I}(b_ic_i)_{i\in I}$$

Theorem (8.1)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- The direct product $\prod_{i \in I} G_i$ is a group.

Proof. The binary operation is associative because

$$\big((a_i)_{i\in I}(b_i)_{i\in I}\big)(c_i)_{i\in I} = (a_ib_i)_{i\in I}(c_i)_{i\in I} = \big((a_ib_i)c_i\big)_{i\in I}$$
$$= \big(a_i(b_ic_i)\big)_{i\in I} = (a_i)_{i\in I}(b_ic_i)_{i\in I} = (a_i)_{i\in I}\big((b_i)_{i\in I}(c_i)_{i\in I}\big).$$

Theorem (8.1)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- The direct product $\prod_{i\in I} G_i$ is a group.

Proof. The binary operation is associative because

$$\big((a_i)_{i\in I}(b_i)_{i\in I}\big)(c_i)_{i\in I} = (a_ib_i)_{i\in I}(c_i)_{i\in I} = \big((a_ib_i)c_i\big)_{i\in I}$$
$$= \big(a_i(b_ic_i)\big)_{i\in I} = (a_i)_{i\in I}(b_ic_i)_{i\in I} = (a_i)_{i\in I}\big((b_i)_{i\in I}(c_i)_{i\in I}\big).$$

Also, $(e_{G_i})_{i\in I}$ is the identity of the direct product, because

Theorem (8.1)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- The direct product $\prod_{i\in I} G_i$ is a group.

Proof. The binary operation is associative because

$$\big((a_i)_{i\in I}(b_i)_{i\in I}\big)(c_i)_{i\in I} = (a_ib_i)_{i\in I}(c_i)_{i\in I} = \big((a_ib_i)c_i\big)_{i\in I}$$

$$= \big(a_i(b_ic_i)\big)_{i\in I} = (a_i)_{i\in I}(b_ic_i)_{i\in I} = (a_i)_{i\in I}\big((b_i)_{i\in I}(c_i)_{i\in I}\big).$$

Also, $(e_{G_i})_{i\in I}$ is the identity of the direct product, because

$$(a_i)_{i\in I}(e_{G_i})_{i\in I}$$

Theorem (8.1)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- The direct product $\prod_{i \in I} G_i$ is a group.

Proof. The binary operation is associative because

$$\big((a_i)_{i\in I}(b_i)_{i\in I}\big)(c_i)_{i\in I} = (a_ib_i)_{i\in I}(c_i)_{i\in I} = \big((a_ib_i)c_i\big)_{i\in I}$$
$$= \big(a_i(b_ic_i)\big)_{i\in I} = (a_i)_{i\in I}(b_ic_i)_{i\in I} = (a_i)_{i\in I}\big((b_i)_{i\in I}(c_i)_{i\in I}\big).$$

Also, $(e_{G_i})_{i\in I}$ is the identity of the direct product, because

$$(a_i)_{i\in I}(e_{G_i})_{i\in I} = (a_ie_{G_i})_{i\in I}$$

Theorem (8.1)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- The direct product $\prod_{i \in I} G_i$ is a group.

Proof. The binary operation is associative because

$$\big((a_i)_{i\in I}(b_i)_{i\in I}\big)(c_i)_{i\in I} = (a_ib_i)_{i\in I}(c_i)_{i\in I} = \big((a_ib_i)c_i\big)_{i\in I}$$
$$= \big(a_i(b_ic_i)\big)_{i\in I} = (a_i)_{i\in I}(b_ic_i)_{i\in I} = (a_i)_{i\in I}\big((b_i)_{i\in I}(c_i)_{i\in I}\big).$$

Also, $(e_{G_i})_{i\in I}$ is the identity of the direct product, because

$$(a_i)_{i\in I}(e_{G_i})_{i\in I} = (a_ie_{G_i})_{i\in I} = (a_i)_{i\in I}$$

Theorem (8.1)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- The direct product $\prod_{i\in I} G_i$ is a group.

Proof. The binary operation is associative because

$$\big((a_i)_{i\in I}(b_i)_{i\in I}\big)(c_i)_{i\in I} = (a_ib_i)_{i\in I}(c_i)_{i\in I} = \big((a_ib_i)c_i\big)_{i\in I}$$
$$= \big(a_i(b_ic_i)\big)_{i\in I} = (a_i)_{i\in I}(b_ic_i)_{i\in I} = (a_i)_{i\in I}\big((b_i)_{i\in I}(c_i)_{i\in I}\big).$$

Also, $(e_{G_i})_{i\in I}$ is the identity of the direct product, because

$$(a_i)_{i\in I}(e_{G_i})_{i\in I} = (a_ie_{G_i})_{i\in I} = (a_i)_{i\in I} \text{ and}$$

Theorem (8.1)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- The direct product $\prod_{i\in I} G_i$ is a group.

Proof. The binary operation is associative because

$$\big((a_i)_{i\in I}(b_i)_{i\in I}\big)(c_i)_{i\in I} = (a_ib_i)_{i\in I}(c_i)_{i\in I} = \big((a_ib_i)c_i\big)_{i\in I}$$
$$= \big(a_i(b_ic_i)\big)_{i\in I} = (a_i)_{i\in I}(b_ic_i)_{i\in I} = (a_i)_{i\in I}\big((b_i)_{i\in I}(c_i)_{i\in I}\big).$$

Also, $(e_{G_i})_{i\in I}$ is the identity of the direct product, because

$(a_i)_{i\in I}(e_{G_i})_{i\in I} = (a_ie_{G_i})_{i\in I} = (a_i)_{i\in I}$ and
$(e_{G_i})_{i\in I}(a_i)_{i\in I}$

Theorem (8.1)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- The direct product $\prod_{i \in I} G_i$ is a group.

Proof. The binary operation is associative because

$$\big((a_i)_{i\in I}(b_i)_{i\in I}\big)(c_i)_{i\in I} = (a_ib_i)_{i\in I}(c_i)_{i\in I} = \big((a_ib_i)c_i\big)_{i\in I}$$
$$= \big(a_i(b_ic_i)\big)_{i\in I} = (a_i)_{i\in I}(b_ic_i)_{i\in I} = (a_i)_{i\in I}\big((b_i)_{i\in I}(c_i)_{i\in I}\big).$$

Also, $(e_{G_i})_{i\in I}$ is the identity of the direct product, because

$$(a_i)_{i\in I}(e_{G_i})_{i\in I} = (a_ie_{G_i})_{i\in I} = (a_i)_{i\in I} \text{ and}$$
$$(e_{G_i})_{i\in I}(a_i)_{i\in I} = (e_{G_i}a_i)_{i\in I}$$

Theorem (8.1)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- The direct product $\prod_{i\in I} G_i$ is a group.

Proof. The binary operation is associative because

$$\big((a_i)_{i\in I}(b_i)_{i\in I}\big)(c_i)_{i\in I} = (a_ib_i)_{i\in I}(c_i)_{i\in I} = \big((a_ib_i)c_i\big)_{i\in I}$$
$$= \big(a_i(b_ic_i)\big)_{i\in I} = (a_i)_{i\in I}(b_ic_i)_{i\in I} = (a_i)_{i\in I}\big((b_i)_{i\in I}(c_i)_{i\in I}\big).$$

Also, $(e_{G_i})_{i\in I}$ is the identity of the direct product, because

$$(a_i)_{i\in I}(e_{G_i})_{i\in I} = (a_ie_{G_i})_{i\in I} = (a_i)_{i\in I} \text{ and}$$
$$(e_{G_i})_{i\in I}(a_i)_{i\in I} = (e_{G_i}a_i)_{i\in I} = (a_i)_{i\in I}.$$

Theorem (8.1)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- The direct product $\prod_{i \in I} G_i$ is a group.

Proof. The binary operation is associative because

$$\big((a_i)_{i\in I}(b_i)_{i\in I}\big)(c_i)_{i\in I} = (a_ib_i)_{i\in I}(c_i)_{i\in I} = \big((a_ib_i)c_i\big)_{i\in I}$$
$$= \big(a_i(b_ic_i)\big)_{i\in I} = (a_i)_{i\in I}(b_ic_i)_{i\in I} = (a_i)_{i\in I}\big((b_i)_{i\in I}(c_i)_{i\in I}\big).$$

Also, $(e_{G_i})_{i\in I}$ is the identity of the direct product, because

$$(a_i)_{i\in I}(e_{G_i})_{i\in I} = (a_ie_{G_i})_{i\in I} = (a_i)_{i\in I} \text{ and}$$
$$(e_{G_i})_{i\in I}(a_i)_{i\in I} = (e_{G_i}a_i)_{i\in I} = (a_i)_{i\in I}.$$

Finally, $(a_i^{-1})_{i\in I}$ is the inverse of $(a_i)_{i\in I}$ because

Theorem (8.1)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- The direct product $\prod_{i \in I} G_i$ is a group.

Proof. The binary operation is associative because

$$\big((a_i)_{i\in I}(b_i)_{i\in I}\big)(c_i)_{i\in I} = (a_ib_i)_{i\in I}(c_i)_{i\in I} = \big((a_ib_i)c_i\big)_{i\in I}$$
$$= \big(a_i(b_ic_i)\big)_{i\in I} = (a_i)_{i\in I}(b_ic_i)_{i\in I} = (a_i)_{i\in I}\big((b_i)_{i\in I}(c_i)_{i\in I}\big).$$

Also, $(e_{G_i})_{i\in I}$ is the identity of the direct product, because

$$(a_i)_{i\in I}(e_{G_i})_{i\in I} = (a_ie_{G_i})_{i\in I} = (a_i)_{i\in I} \text{ and}$$
$$(e_{G_i})_{i\in I}(a_i)_{i\in I} = (e_{G_i}a_i)_{i\in I} = (a_i)_{i\in I}.$$

Finally, $(a_i^{-1})_{i\in I}$ is the inverse of $(a_i)_{i\in I}$ because

$$(a_i)_{i\in I}(a_i^{-1})_{i\in I}$$

Theorem (8.1)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- The direct product $\prod_{i\in I} G_i$ is a group.

Proof. The binary operation is associative because

$$\big((a_i)_{i\in I}(b_i)_{i\in I}\big)(c_i)_{i\in I} = (a_ib_i)_{i\in I}(c_i)_{i\in I} = \big((a_ib_i)c_i\big)_{i\in I}$$
$$= \big(a_i(b_ic_i)\big)_{i\in I} = (a_i)_{i\in I}(b_ic_i)_{i\in I} = (a_i)_{i\in I}\big((b_i)_{i\in I}(c_i)_{i\in I}\big).$$

Also, $(e_{G_i})_{i\in I}$ is the identity of the direct product, because

$$(a_i)_{i\in I}(e_{G_i})_{i\in I} = (a_ie_{G_i})_{i\in I} = (a_i)_{i\in I} \text{ and}$$
$$(e_{G_i})_{i\in I}(a_i)_{i\in I} = (e_{G_i}a_i)_{i\in I} = (a_i)_{i\in I}.$$

Finally, $(a_i^{-1})_{i\in I}$ is the inverse of $(a_i)_{i\in I}$ because

$$(a_i)_{i\in I}(a_i^{-1})_{i\in I} = (a_ia_i^{-1})_{i\in I}$$

Theorem (8.1)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- The direct product $\prod_{i\in I} G_i$ is a group.

Proof. The binary operation is associative because

$$\big((a_i)_{i\in I}(b_i)_{i\in I}\big)(c_i)_{i\in I} = (a_ib_i)_{i\in I}(c_i)_{i\in I} = \big((a_ib_i)c_i\big)_{i\in I}$$
$$= \big(a_i(b_ic_i)\big)_{i\in I} = (a_i)_{i\in I}(b_ic_i)_{i\in I} = (a_i)_{i\in I}\big((b_i)_{i\in I}(c_i)_{i\in I}\big).$$

Also, $(e_{G_i})_{i\in I}$ is the identity of the direct product, because

$$(a_i)_{i\in I}(e_{G_i})_{i\in I} = (a_ie_{G_i})_{i\in I} = (a_i)_{i\in I} \text{ and}$$
$$(e_{G_i})_{i\in I}(a_i)_{i\in I} = (e_{G_i}a_i)_{i\in I} = (a_i)_{i\in I}.$$

Finally, $(a_i^{-1})_{i\in I}$ is the inverse of $(a_i)_{i\in I}$ because

$$(a_i)_{i\in I}(a_i^{-1})_{i\in I} = (a_ia_i^{-1})_{i\in I} = (e_{G_i})_{i\in I}$$

Theorem (8.1)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- The direct product $\prod_{i\in I} G_i$ is a group.

Proof. The binary operation is associative because

$$\big((a_i)_{i\in I}(b_i)_{i\in I}\big)(c_i)_{i\in I} = (a_ib_i)_{i\in I}(c_i)_{i\in I} = \big((a_ib_i)c_i\big)_{i\in I}$$
$$= \big(a_i(b_ic_i)\big)_{i\in I} = (a_i)_{i\in I}(b_ic_i)_{i\in I} = (a_i)_{i\in I}\big((b_i)_{i\in I}(c_i)_{i\in I}\big).$$

Also, $(e_{G_i})_{i\in I}$ is the identity of the direct product, because

$$(a_i)_{i\in I}(e_{G_i})_{i\in I} = (a_ie_{G_i})_{i\in I} = (a_i)_{i\in I} \text{ and}$$
$$(e_{G_i})_{i\in I}(a_i)_{i\in I} = (e_{G_i}a_i)_{i\in I} = (a_i)_{i\in I}.$$

Finally, $(a_i^{-1})_{i\in I}$ is the inverse of $(a_i)_{i\in I}$ because

$$(a_i)_{i\in I}(a_i^{-1})_{i\in I} = (a_ia_i^{-1})_{i\in I} = (e_{G_i})_{i\in I} \text{ and}$$
$$(a_i^{-1})_{i\in I}(a_i)_{i\in I}$$

Theorem (8.1)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- The direct product $\prod_{i\in I} G_i$ is a group.

Proof. The binary operation is associative because

$$\big((a_i)_{i\in I}(b_i)_{i\in I}\big)(c_i)_{i\in I} = (a_ib_i)_{i\in I}(c_i)_{i\in I} = \big((a_ib_i)c_i\big)_{i\in I}$$
$$= \big(a_i(b_ic_i)\big)_{i\in I} = (a_i)_{i\in I}(b_ic_i)_{i\in I} = (a_i)_{i\in I}\big((b_i)_{i\in I}(c_i)_{i\in I}\big).$$

Also, $(e_{G_i})_{i\in I}$ is the identity of the direct product, because

$$(a_i)_{i\in I}(e_{G_i})_{i\in I} = (a_ie_{G_i})_{i\in I} = (a_i)_{i\in I} \text{ and}$$
$$(e_{G_i})_{i\in I}(a_i)_{i\in I} = (e_{G_i}a_i)_{i\in I} = (a_i)_{i\in I}.$$

Finally, $(a_i^{-1})_{i\in I}$ is the inverse of $(a_i)_{i\in I}$ because

$$(a_i)_{i\in I}(a_i^{-1})_{i\in I} = (a_ia_i^{-1})_{i\in I} = (e_{G_i})_{i\in I} \text{ and}$$
$$(a_i^{-1})_{i\in I}(a_i)_{i\in I} = (a_i^{-1}a_i)_{i\in I}$$

Theorem (8.1)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- The direct product $\prod_{i\in I} G_i$ is a group.

Proof. The binary operation is associative because

$$\big((a_i)_{i\in I}(b_i)_{i\in I}\big)(c_i)_{i\in I} = (a_ib_i)_{i\in I}(c_i)_{i\in I} = \big((a_ib_i)c_i\big)_{i\in I}$$
$$= \big(a_i(b_ic_i)\big)_{i\in I} = (a_i)_{i\in I}(b_ic_i)_{i\in I} = (a_i)_{i\in I}\big((b_i)_{i\in I}(c_i)_{i\in I}\big).$$

Also, $(e_{G_i})_{i\in I}$ is the identity of the direct product, because

$$(a_i)_{i\in I}(e_{G_i})_{i\in I} = (a_ie_{G_i})_{i\in I} = (a_i)_{i\in I} \text{ and}$$
$$(e_{G_i})_{i\in I}(a_i)_{i\in I} = (e_{G_i}a_i)_{i\in I} = (a_i)_{i\in I}.$$

Finally, $(a_i^{-1})_{i\in I}$ is the inverse of $(a_i)_{i\in I}$ because

$$(a_i)_{i\in I}(a_i^{-1})_{i\in I} = (a_ia_i^{-1})_{i\in I} = (e_{G_i})_{i\in I} \text{ and}$$
$$(a_i^{-1})_{i\in I}(a_i)_{i\in I} = (a_i^{-1}a_i)_{i\in I} = (e_{G_i})_{i\in I}.$$

Theorem (8.1)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- The direct product $\prod_{i\in I} G_i$ is a group.
- For each $k \in I$,

Proof. The binary operation is associative because

$$\big((a_i)_{i\in I}(b_i)_{i\in I}\big)(c_i)_{i\in I} = (a_ib_i)_{i\in I}(c_i)_{i\in I} = \big((a_ib_i)c_i\big)_{i\in I}$$
$$= \big(a_i(b_ic_i)\big)_{i\in I} = (a_i)_{i\in I}(b_ic_i)_{i\in I} = (a_i)_{i\in I}\big((b_i)_{i\in I}(c_i)_{i\in I}\big).$$

Also, $(e_{G_i})_{i\in I}$ is the identity of the direct product, because

$$(a_i)_{i\in I}(e_{G_i})_{i\in I} = (a_ie_{G_i})_{i\in I} = (a_i)_{i\in I} \text{ and}$$
$$(e_{G_i})_{i\in I}(a_i)_{i\in I} = (e_{G_i}a_i)_{i\in I} = (a_i)_{i\in I}.$$

Finally, $(a_i^{-1})_{i\in I}$ is the inverse of $(a_i)_{i\in I}$ because

$$(a_i)_{i\in I}(a_i^{-1})_{i\in I} = (a_ia_i^{-1})_{i\in I} = (e_{G_i})_{i\in I} \text{ and}$$
$$(a_i^{-1})_{i\in I}(a_i)_{i\in I} = (a_i^{-1}a_i)_{i\in I} = (e_{G_i})_{i\in I}.$$

Theorem (8.1)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- The direct product $\prod_{i\in I} G_i$ is a group.
- For each $k \in I$, the canonical projection $\pi_k : \prod_{i\in I} G_i \to G_k$

Proof. The binary operation is associative because

$$\big((a_i)_{i\in I}(b_i)_{i\in I}\big)(c_i)_{i\in I} = (a_ib_i)_{i\in I}(c_i)_{i\in I} = \big((a_ib_i)c_i\big)_{i\in I}$$
$$= \big(a_i(b_ic_i)\big)_{i\in I} = (a_i)_{i\in I}(b_ic_i)_{i\in I} = (a_i)_{i\in I}\big((b_i)_{i\in I}(c_i)_{i\in I}\big).$$

Also, $(e_{G_i})_{i\in I}$ is the identity of the direct product, because

$$(a_i)_{i\in I}(e_{G_i})_{i\in I} = (a_ie_{G_i})_{i\in I} = (a_i)_{i\in I} \text{ and}$$
$$(e_{G_i})_{i\in I}(a_i)_{i\in I} = (e_{G_i}a_i)_{i\in I} = (a_i)_{i\in I}.$$

Finally, $(a_i^{-1})_{i\in I}$ is the inverse of $(a_i)_{i\in I}$ because

$$(a_i)_{i\in I}(a_i^{-1})_{i\in I} = (a_ia_i^{-1})_{i\in I} = (e_{G_i})_{i\in I} \text{ and}$$
$$(a_i^{-1})_{i\in I}(a_i)_{i\in I} = (a_i^{-1}a_i)_{i\in I} = (e_{G_i})_{i\in I}.$$

Theorem (8.1)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- The direct product $\prod_{i\in I} G_i$ is a group.
- For each $k \in I$, the canonical projection $\pi_k : \prod_{i\in I} G_i \to G_k$ is an epimorphism of groups.

Proof. The binary operation is associative because

$$\big((a_i)_{i\in I}(b_i)_{i\in I}\big)(c_i)_{i\in I} = (a_ib_i)_{i\in I}(c_i)_{i\in I} = \big((a_ib_i)c_i\big)_{i\in I}$$
$$= \big(a_i(b_ic_i)\big)_{i\in I} = (a_i)_{i\in I}(b_ic_i)_{i\in I} = (a_i)_{i\in I}\big((b_i)_{i\in I}(c_i)_{i\in I}\big).$$

Also, $(e_{G_i})_{i\in I}$ is the identity of the direct product, because

$$(a_i)_{i\in I}(e_{G_i})_{i\in I} = (a_ie_{G_i})_{i\in I} = (a_i)_{i\in I} \text{ and}$$
$$(e_{G_i})_{i\in I}(a_i)_{i\in I} = (e_{G_i}a_i)_{i\in I} = (a_i)_{i\in I}.$$

Finally, $(a_i^{-1})_{i\in I}$ is the inverse of $(a_i)_{i\in I}$ because

$$(a_i)_{i\in I}(a_i^{-1})_{i\in I} = (a_ia_i^{-1})_{i\in I} = (e_{G_i})_{i\in I} \text{ and}$$
$$(a_i^{-1})_{i\in I}(a_i)_{i\in I} = (a_i^{-1}a_i)_{i\in I} = (e_{G_i})_{i\in I}.$$

Theorem (8.1)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- The direct product $\prod_{i\in I} G_i$ is a group.
- For each $k \in I$, the canonical projection $\pi_k : \prod_{i\in I} G_i \to G_k$ is an epimorphism of groups.

Proof.

Theorem (8.1)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- The direct product $\prod_{i \in I} G_i$ is a group.
- For each $k \in I$, the canonical projection $\pi_k : \prod_{i \in I} G_i \to G_k$ is an epimorphism of groups.

Proof. π_k is a group homomorphism because

Theorem (8.1)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- The direct product $\prod_{i\in I} G_i$ is a group.
- For each $k \in I$, the canonical projection $\pi_k : \prod_{i\in I} G_i \to G_k$ is an epimorphism of groups.

Proof. π_k is a group homomorphism because

$$\pi_k\big((a_i)_{i\in I}(b_i)_{i\in I}\big)$$

Theorem (8.1)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- The direct product $\prod_{i \in I} G_i$ is a group.
- For each $k \in I$, the canonical projection $\pi_k : \prod_{i \in I} G_i \to G_k$ is an epimorphism of groups.

Proof. π_k is a group homomorphism because

$$\pi_k\big((a_i)_{i \in I}(b_i)_{i \in I}\big) \;=\; \pi_k\big((a_i b_i)_{i \in I}\big)$$

Theorem (8.1)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- The direct product $\prod_{i\in I} G_i$ is a group.
- For each $k \in I$, the canonical projection $\pi_k : \prod_{i\in I} G_i \to G_k$ is an epimorphism of groups.

Proof. π_k is a group homomorphism because

$$\begin{aligned} \pi_k\Big((a_i)_{i\in I}(b_i)_{i\in I}\Big) &= \pi_k\Big((a_ib_i)_{i\in I}\Big) \\ &= a_kb_k \end{aligned}$$

Theorem (8.1)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- The direct product $\prod_{i\in I} G_i$ is a group.
- For each $k \in I$, the canonical projection $\pi_k : \prod_{i\in I} G_i \to G_k$ is an epimorphism of groups.

Proof. π_k is a group homomorphism because

$$\begin{aligned} \pi_k\big((a_i)_{i\in I}(b_i)_{i\in I}\big) &= \pi_k\big((a_ib_i)_{i\in I}\big) \\ &= a_kb_k \\ &= \pi_k\big((a_i)_{i\in I}\big)\pi_k\big((b_i)_{i\in I}\big). \end{aligned}$$

Theorem (8.1)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- The direct product $\prod_{i\in I} G_i$ is a group.
- For each $k \in I$, the canonical projection $\pi_k : \prod_{i\in I} G_i \to G_k$ is an epimorphism of groups.

Proof. π_k is a group homomorphism because

$$\begin{aligned}
\pi_k\big((a_i)_{i\in I}(b_i)_{i\in I}\big) &= \pi_k\big((a_ib_i)_{i\in I}\big) \\
&= a_kb_k \\
&= \pi_k\big((a_i)_{i\in I}\big)\pi_k\big((b_i)_{i\in I}\big).
\end{aligned}$$

Moreover, π_k is onto, because

Theorem (8.1)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- The direct product $\prod_{i \in I} G_i$ is a group.
- For each $k \in I$, the canonical projection $\pi_k : \prod_{i \in I} G_i \to G_k$ is an epimorphism of groups.

Proof. π_k is a group homomorphism because

$$\begin{aligned} \pi_k\big((a_i)_{i\in I}(b_i)_{i\in I}\big) &= \pi_k\big((a_i b_i)_{i\in I}\big) \\ &= a_k b_k \\ &= \pi_k\big((a_i)_{i\in I}\big)\pi_k\big((b_i)_{i\in I}\big). \end{aligned}$$

Moreover, π_k is onto, because every G_i is nonempty (since G_i contains at least the identity element).

Theorem (8.1)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- The direct product $\prod_{i \in I} G_i$ is a group.
- For each $k \in I$, the canonical projection $\pi_k : \prod_{i \in I} G_i \to G_k$ is an epimorphism of groups.

Proof. π_k is a group homomorphism because

$$\begin{aligned} \pi_k\big((a_i)_{i\in I}(b_i)_{i\in I}\big) &= \pi_k\big((a_i b_i)_{i\in I}\big) \\ &= a_k b_k \\ &= \pi_k\big((a_i)_{i\in I}\big)\pi_k\big((b_i)_{i\in I}\big). \end{aligned}$$

Moreover, π_k is onto, because every G_i is nonempty (since G_i contains at least the identity element).

Theorem (8.2)

Let $\{G_i \mid i \in I\}$ be a family of groups.

Theorem (8.2)

Let $\{G_i \mid i \in I\}$ be a family of groups. Suppose $\{\varphi_i : H \to G_i \mid i \in I\}$ is a family of group homomorphisms.

Theorem (8.2)

Let $\{G_i \mid i \in I\}$ be a family of groups. Suppose $\{\varphi_i : H \to G_i \mid i \in I\}$ is a family of group homomorphisms. Then there exists a unique homomorphism $\varphi : H \to \prod_{i \in I} G_i$

Theorem (8.2)

Let $\{G_i \mid i \in I\}$ be a family of groups. Suppose $\{\varphi_i : H \to G_i \mid i \in I\}$ is a family of group homomorphisms. Then there exists a unique homomorphism $\varphi : H \to \prod_{i \in I} G_i$ such that $\pi_i \varphi = \varphi_i$ for all $i \in I$,

Theorem (8.2)

Let $\{G_i \mid i \in I\}$ be a family of groups. Suppose $\{\varphi_i : H \to G_i \mid i \in I\}$ is a family of group homomorphisms. Then there exists a unique homomorphism $\varphi : H \to \prod_{i \in I} G_i$ such that $\pi_i \varphi = \varphi_i$ for all $i \in I$, where $\pi_i : \prod_{i \in I} G_i \to G_i$ is the canonical projection.

Theorem (8.2)

Let $\{G_i \mid i \in I\}$ be a family of groups. Suppose $\{\varphi_i : H \to G_i \mid i \in I\}$ is a family of group homomorphisms. Then there exists a unique homomorphism $\varphi : H \to \prod_{i \in I} G_i$ such that $\pi_i \varphi = \varphi_i$ for all $i \in I$, where $\pi_i : \prod_{i \in I} G_i \to G_i$ is the canonical projection. Moreover, this property determines $\prod_{i \in I} G_i$ uniquely up to isomorphism.

Theorem (8.2)

Let $\{G_i \mid i \in I\}$ be a family of groups. Suppose $\{\varphi_i : H \to G_i \mid i \in I\}$ is a family of group homomorphisms. Then there exists a unique homomorphism $\varphi : H \to \prod_{i\in I} G_i$ such that $\pi_i \varphi = \varphi_i$ for all $i \in I$, where $\pi_i : \prod_{i\in I} G_i \to G_i$ is the canonical projection. Moreover, this property determines $\prod_{i\in I} G_i$ uniquely up to isomorphism. In particular,

Theorem (8.2)

Let $\{G_i \mid i \in I\}$ be a family of groups. Suppose $\{\varphi_i : H \to G_i \mid i \in I\}$ is a family of group homomorphisms. Then there exists a unique homomorphism $\varphi : H \to \prod_{i \in I} G_i$ such that $\pi_i \varphi = \varphi_i$ for all $i \in I$, where $\pi_i : \prod_{i \in I} G_i \to G_i$ is the canonical projection. Moreover, this property determines $\prod_{i \in I} G_i$ uniquely up to isomorphism. In particular, $\prod_{i \in I} G_i$ is a product in the category $\mathcal{G}$ of groups.

Theorem (8.2)

Let $\{G_i \mid i \in I\}$ be a family of groups. Suppose $\{\varphi_i : H \to G_i \mid i \in I\}$ is a family of group homomorphisms. Then there exists a unique homomorphism $\varphi : H \to \prod_{i \in I} G_i$ such that $\pi_i \varphi = \varphi_i$ for all $i \in I$, where $\pi_i : \prod_{i \in I} G_i \to G_i$ is the canonical projection. Moreover, this property determines $\prod_{i \in I} G_i$ uniquely up to isomorphism. In particular, $\prod_{i \in I} G_i$ is a product in the category $\mathcal{G}$ of groups.

Proof. In the proof of Theorem 5.2 in Introduction, we show that the map $\varphi : H \to \prod_{i \in I} G_i$ defined by $\varphi(a) = \big(\varphi_i(a)\big)_{i \in I}$ is the unique function such that $\pi_i \varphi = \varphi_i \; \forall i \in I$.

Theorem (8.2)

Let $\{G_i \mid i \in I\}$ be a family of groups. Suppose $\{\varphi_i : H \to G_i \mid i \in I\}$ is a family of group homomorphisms. Then there exists a unique homomorphism $\varphi : H \to \prod_{i\in I} G_i$ such that $\pi_i\varphi = \varphi_i$ for all $i \in I$, where $\pi_i : \prod_{i\in I} G_i \to G_i$ is the canonical projection. Moreover, this property determines $\prod_{i\in I} G_i$ uniquely up to isomorphism. In particular, $\prod_{i\in I} G_i$ is a product in the category $\mathcal{G}$ of groups.

Proof. In the proof of Theorem 5.2 in Introduction, we show that the map $\varphi : H \to \prod_{i\in I} G_i$ defined by $\varphi(a) = \big(\varphi_i(a)\big)_{i\in I}$ is the unique function such that $\pi_i\varphi = \varphi_i$ $\forall i \in I$.

Theorem (8.2)

Let $\{G_i \mid i \in I\}$ be a family of groups. Suppose $\{\varphi_i : H \to G_i \mid i \in I\}$ is a family of group homomorphisms. Then there exists a unique homomorphism $\varphi : H \longrightarrow \prod_{i \in I} G_i$ such that $\pi_i \varphi = \varphi_i$ for all $i \in I$, where $\pi_i : \prod_{i \in I} G_i \to G_i$ is the canonical projection. Moreover, this property determines $\prod_{i \in I} G_i$ uniquely up to isomorphism. In particular, $\prod_{i \in I} G_i$ is a product in the category $\mathcal{G}$ of groups.

Proof. In the proof of Theorem 5.2 in Introduction, we show that the map $\varphi : H \to \prod_{i \in I} G_i$ defined by $\varphi(a) = \big(\varphi_i(a)\big)_{i \in I}$ is the unique function such that $\pi_i \varphi = \varphi_i \ \forall i \in I$.

Theorem (8.2)

Let $\{G_i \mid i \in I\}$ be a family of groups. Suppose $\{\varphi_i : H \to G_i \mid i \in I\}$ is a family of group homomorphisms. Then there exists a unique homomorphism $\varphi : H \to \prod_{i \in I} G_i$ such that $\pi_i \varphi = \varphi_i$ for all $i \in I$, where $\pi_i : \prod_{i \in I} G_i \to G_i$ is the canonical projection. Moreover, this property determines $\prod_{i \in I} G_i$ uniquely up to isomorphism. In particular, $\prod_{i \in I} G_i$ is a product in the category $\mathcal{G}$ of groups.

Proof. In the proof of Theorem 5.2 in Introduction, we show that the map $\varphi : H \to \prod_{i \in I} G_i$ defined by $\varphi(a) = \big(\varphi_i(a)\big)_{i \in I}$ is the unique function such that $\pi_i \varphi = \varphi_i \; \forall i \in I$.

Theorem (8.2)

Let $\{G_i \mid i \in I\}$ be a family of groups. Suppose $\{\varphi_i : H \to G_i \mid i \in I\}$ is a family of group homomorphisms. Then there exists a unique homomorphism $\varphi : H \to \prod_{i \in I} G_i$ such that $\pi_i \varphi = \varphi_i$ for all $i \in I$, where $\pi_i : \prod_{i \in I} G_i \to G_i$ is the canonical projection. Moreover, this property determines $\prod_{i \in I} G_i$ uniquely up to isomorphism. In particular, $\prod_{i \in I} G_i$ is a product in the category $\mathcal{G}$ of groups.

Proof. In the proof of Theorem 5.2 in Introduction, we show that the map $\varphi : H \to \prod_{i \in I} G_i$ defined by $\varphi(a) = \big(\varphi_i(a)\big)_{i \in I}$ is the unique function such that $\pi_i \varphi = \varphi_i \; \forall i \in I$. It is easy to verify that φ is a homomorphism of groups.

Theorem (8.2)

Let $\{G_i \mid i \in I\}$ be a family of groups. Suppose $\{\varphi_i : H \to G_i \mid i \in I\}$ is a family of group homomorphisms. Then there exists a unique homomorphism $\varphi : H \to \prod_{i\in I} G_i$ such that $\pi_i\varphi = \varphi_i$ for all $i \in I$, where $\pi_i : \prod_{i\in I} G_i \to G_i$ is the canonical projection. Moreover, this property determines $\prod_{i\in I} G_i$ uniquely up to isomorphism. In particular, $\prod_{i\in I} G_i$ is a product in the category $\mathcal{G}$ of groups.

Proof. In the proof of Theorem 5.2 in Introduction, we show that the map $\varphi : H \to \prod_{i\in I} G_i$ defined by $\varphi(a) = \big(\varphi_i(a)\big)_{i\in I}$ is the unique function such that $\pi_i\varphi = \varphi_i\ \forall i \in I$. It is easy to verify that φ is a homomorphism of groups. Indeed,

Theorem (8.2)

Let $\{G_i \mid i \in I\}$ be a family of groups. Suppose $\{\varphi_i : H \to G_i \mid i \in I\}$ is a family of group homomorphisms. Then there exists a unique homomorphism $\varphi : H \to \prod_{i\in I} G_i$ such that $\pi_i\varphi = \varphi_i$ for all $i \in I$, where $\pi_i : \prod_{i\in I} G_i \to G_i$ is the canonical projection. Moreover, this property determines $\prod_{i\in I} G_i$ uniquely up to isomorphism. In particular, $\prod_{i\in I} G_i$ is a product in the category $\mathcal{G}$ of groups.

Proof. In the proof of Theorem 5.2 in Introduction, we show that the map $\varphi : H \to \prod_{i\in I} G_i$ defined by $\varphi(a) = \big(\varphi_i(a)\big)_{i\in I}$ is the unique function such that $\pi_i\varphi = \varphi_i \ \forall i \in I$. It is easy to verify that φ is a homomorphism of groups. Indeed, $\varphi(ab)$

Theorem (8.2)

Let $\{G_i \mid i \in I\}$ be a family of groups. Suppose $\{\varphi_i : H \to G_i \mid i \in I\}$ is a family of group homomorphisms. Then there exists a unique homomorphism $\varphi : H \to \prod_{i \in I} G_i$ such that $\pi_i \varphi = \varphi_i$ for all $i \in I$, where $\pi_i : \prod_{i \in I} G_i \to G_i$ is the canonical projection. Moreover, this property determines $\prod_{i \in I} G_i$ uniquely up to isomorphism. In particular, $\prod_{i \in I} G_i$ is a product in the category $\mathcal{G}$ of groups.

Proof. In the proof of Theorem 5.2 in Introduction, we show that the map $\varphi : H \to \prod_{i \in I} G_i$ defined by $\varphi(a) = \big(\varphi_i(a)\big)_{i \in I}$ is the unique function such that $\pi_i \varphi = \varphi_i \ \forall i \in I$. It is easy to verify that φ is a homomorphism of groups. Indeed, $\varphi(ab)$
$= \big(\varphi_i(ab)\big)_{i \in I}$

Theorem (8.2)

Let $\{G_i \mid i \in I\}$ be a family of groups. Suppose $\{\varphi_i : H \to G_i \mid i \in I\}$ is a family of group homomorphisms. Then there exists a unique homomorphism $\varphi : H \to \prod_{i\in I} G_i$ such that $\pi_i\varphi = \varphi_i$ for all $i \in I$, where $\pi_i : \prod_{i\in I} G_i \to G_i$ is the canonical projection. Moreover, this property determines $\prod_{i\in I} G_i$ uniquely up to isomorphism. In particular, $\prod_{i\in I} G_i$ is a product in the category $\mathcal{G}$ of groups.

Proof. In the proof of Theorem 5.2 in Introduction, we show that the map $\varphi : H \to \prod_{i\in I} G_i$ defined by $\varphi(a) = \big(\varphi_i(a)\big)_{i\in I}$ is the unique function such that $\pi_i\varphi = \varphi_i \; \forall i \in I$. It is easy to verify that φ is a homomorphism of groups. Indeed, $\varphi(ab)$ $= \big(\varphi_i(ab)\big)_{i\in I} = \big(\varphi_i(a)\varphi_i(b)\big)_{i\in I}$

Theorem (8.2)

Let $\{G_i \mid i \in I\}$ be a family of groups. Suppose $\{\varphi_i : H \to G_i \mid i \in I\}$ is a family of group homomorphisms. Then there exists a unique homomorphism $\varphi : H \to \prod_{i\in I} G_i$ such that $\pi_i\varphi = \varphi_i$ for all $i \in I$, where $\pi_i : \prod_{i\in I} G_i \to G_i$ is the canonical projection. Moreover, this property determines $\prod_{i\in I} G_i$ uniquely up to isomorphism. In particular, $\prod_{i\in I} G_i$ is a product in the category $\mathcal{G}$ of groups.

Proof. In the proof of Theorem 5.2 in Introduction, we show that the map $\varphi : H \to \prod_{i\in I} G_i$ defined by $\varphi(a) = \big(\varphi_i(a)\big)_{i\in I}$ is the unique function such that $\pi_i\varphi = \varphi_i$ $\forall i \in I$. It is easy to verify that φ is a homomorphism of groups. Indeed, $\varphi(ab)$ $= \big(\varphi_i(ab)\big)_{i\in I} = \big(\varphi_i(a)\varphi_i(b)\big)_{i\in I} = \big(\varphi_i(a)\big)_{i\in I}\big(\varphi_i(b)\big)_{i\in I}$

Theorem (8.2)

Let $\{G_i \mid i \in I\}$ be a family of groups. Suppose $\{\varphi_i : H \to G_i \mid i \in I\}$ is a family of group homomorphisms. Then there exists a unique homomorphism $\varphi : H \to \prod_{i \in I} G_i$ such that $\pi_i \varphi = \varphi_i$ for all $i \in I$, where $\pi_i : \prod_{i \in I} G_i \to G_i$ is the canonical projection. Moreover, this property determines $\prod_{i \in I} G_i$ uniquely up to isomorphism. In particular, $\prod_{i \in I} G_i$ is a product in the category $\mathcal{G}$ of groups.

Proof. In the proof of Theorem 5.2 in Introduction, we show that the map $\varphi : H \to \prod_{i \in I} G_i$ defined by $\varphi(a) = \big(\varphi_i(a)\big)_{i \in I}$ is the unique function such that $\pi_i \varphi = \varphi_i \ \forall i \in I$. It is easy to verify that φ is a homomorphism of groups. Indeed, $\varphi(ab)$
$= \big(\varphi_i(ab)\big)_{i \in I} = \big(\varphi_i(a)\varphi_i(b)\big)_{i \in I} = \big(\varphi_i(a)\big)_{i \in I}\big(\varphi_i(b)\big)_{i \in I}$
$= \varphi(a)\varphi(b)$.

Theorem (8.2)

Let $\{G_i \mid i \in I\}$ be a family of groups. Suppose $\{\varphi_i : H \to G_i \mid i \in I\}$ is a family of group homomorphisms. Then there exists a unique homomorphism $\varphi : H \to \prod_{i\in I} G_i$ such that $\pi_i\varphi = \varphi_i$ for all $i \in I$, where $\pi_i : \prod_{i\in I} G_i \to G_i$ is the canonical projection. Moreover, this property determines $\prod_{i\in I} G_i$ uniquely up to isomorphism. In particular, $\prod_{i\in I} G_i$ is a product in the category $\mathcal{G}$ of groups.

Proof. In the proof of Theorem 5.2 in Introduction, we show that the map $\varphi : H \to \prod_{i\in I} G_i$ defined by $\varphi(a) = \big(\varphi_i(a)\big)_{i\in I}$ is the unique function such that $\pi_i\varphi = \varphi_i$ $\forall i \in I$. It is easy to verify that φ is a homomorphism of groups. Indeed, $\varphi(ab)$
$= \big(\varphi_i(ab)\big)_{i\in I} = \big(\varphi_i(a)\varphi_i(b)\big)_{i\in I} = \big(\varphi_i(a)\big)_{i\in I}\big(\varphi_i(b)\big)_{i\in I}$
$= \varphi(a)\varphi(b)$. Therefore, we finish proving the statement:

Theorem (8.2)

Let $\{G_i \mid i \in I\}$ be a family of groups. Suppose $\{\varphi_i : H \to G_i \mid i \in I\}$ is a family of group homomorphisms. Then there exists a unique homomorphism $\varphi : H \to \prod_{i \in I} G_i$ such that $\pi_i \varphi = \varphi_i$ for all $i \in I$, where $\pi_i : \prod_{i \in I} G_i \to G_i$ is the canonical projection. Moreover, this property determines $\prod_{i \in I} G_i$ uniquely up to isomorphism. In particular, $\prod_{i \in I} G_i$ is a product in the category $\mathcal{G}$ of groups.

Proof. In the proof of Theorem 5.2 in Introduction, we show that the map $\varphi : H \to \prod_{i \in I} G_i$ defined by $\varphi(a) = \big(\varphi_i(a)\big)_{i \in I}$ is the unique function such that $\pi_i \varphi = \varphi_i$ $\forall i \in I$. It is easy to verify that φ is a homomorphism of groups. Indeed, $\varphi(ab)$ $= \big(\varphi_i(ab)\big)_{i \in I} = \big(\varphi_i(a)\varphi_i(b)\big)_{i \in I} = \big(\varphi_i(a)\big)_{i \in I}\big(\varphi_i(b)\big)_{i \in I}$ $= \varphi(a)\varphi(b)$. Therefore, we finish proving the statement:

Theorem (8.2)

Let $\{G_i \mid i \in I\}$ be a family of groups. Suppose $\{\varphi_i : H \to G_i \mid i \in I\}$ is a family of group homomorphisms. Then there exists a unique homomorphism $\varphi : H \to \prod_{i \in I} G_i$ such that $\pi_i \varphi = \varphi_i$ for all $i \in I$, where $\pi_i : \prod_{i \in I} G_i \to G_i$ is the canonical projection. Moreover, this property determines $\prod_{i \in I} G_i$ uniquely up to isomorphism. In particular, $\prod_{i \in I} G_i$ is a product in the category $\mathcal{G}$ of groups.

Proof. Hence, in the category $\mathcal{G}$ of groups,

Theorem (8.2)

Let $\{G_i \mid i \in I\}$ be a family of groups. Suppose $\{\varphi_i : H \to G_i \mid i \in I\}$ is a family of group homomorphisms. Then there exists a unique homomorphism $\varphi : H \to \prod_{i \in I} G_i$ such that $\pi_i \varphi = \varphi_i$ for all $i \in I$, where $\pi_i : \prod_{i \in I} G_i \to G_i$ is the canonical projection. Moreover, this property determines $\prod_{i \in I} G_i$ uniquely up to isomorphism. In particular, $\prod_{i \in I} G_i$ is a product in the category $\mathcal{G}$ of groups.

Proof. Hence, in the category $\mathcal{G}$ of groups, $\prod_{i \in I} G_i$ is a product.

Theorem (8.2)

Let $\{G_i \mid i \in I\}$ be a family of groups. Suppose $\{\varphi_i : H \to G_i \mid i \in I\}$ is a family of group homomorphisms. Then there exists a unique homomorphism $\varphi : H \to \prod_{i \in I} G_i$ such that $\pi_i \varphi = \varphi_i$ for all $i \in I$, where $\pi_i : \prod_{i \in I} G_i \to G_i$ is the canonical projection. Moreover, this property determines $\prod_{i \in I} G_i$ uniquely up to isomorphism. In particular, $\prod_{i \in I} G_i$ is a product in the category $\mathcal{G}$ of groups.

Proof. Hence, in the category $\mathcal{G}$ of groups, $\prod_{i \in I} G_i$ is a product. Since in this category $\mathcal{G}$,

Theorem (8.2)

Let $\{G_i \mid i \in I\}$ be a family of groups. Suppose $\{\varphi_i : H \to G_i \mid i \in I\}$ is a family of group homomorphisms. Then there exists a unique homomorphism $\varphi : H \to \prod_{i \in I} G_i$ such that $\pi_i \varphi = \varphi_i$ for all $i \in I$, where $\pi_i : \prod_{i \in I} G_i \to G_i$ is the canonical projection. Moreover, this property determines $\prod_{i \in I} G_i$ uniquely up to isomorphism. In particular, $\prod_{i \in I} G_i$ is a product in the category $\mathcal{G}$ of groups.

Proof. Hence, in the category $\mathcal{G}$ of groups, $\prod_{i \in I} G_i$ is a product. Since in this category $\mathcal{G}$, an equivalence is an isomorphism of groups,

Theorem (8.2)

Let $\{G_i \mid i \in I\}$ be a family of groups. Suppose $\{\varphi_i : H \to G_i \mid i \in I\}$ is a family of group homomorphisms. Then there exists a unique homomorphism $\varphi : H \to \prod_{i \in I} G_i$ such that $\pi_i \varphi = \varphi_i$ for all $i \in I$, where $\pi_i : \prod_{i \in I} G_i \to G_i$ is the canonical projection. Moreover, this property determines $\prod_{i \in I} G_i$ uniquely up to isomorphism. In particular, $\prod_{i \in I} G_i$ is a product in the category $\mathcal{G}$ of groups.

Proof. Hence, in the category $\mathcal{G}$ of groups, $\prod_{i \in I} G_i$ is a product. Since in this category $\mathcal{G}$, an equivalence is an isomorphism of groups, by Theorem (7.3), $\prod_{i \in I} G_i$ is uniquely determined up to isomorphism.

Weak Direct Products

Definition (8.3). Let $\{G_i \mid i \in I\}$ be a family of groups.

Weak Direct Products

Definition (8.3)**.** Let $\{G_i \mid i \in I\}$ be a family of groups.

- The (external) weak direct product of $\{G_i \mid i \in I\}$,

Weak Direct Products

Definition (8.3). Let $\{G_i \mid i \in I\}$ be a family of groups.

- The (external) weak direct product of $\{G_i \mid i \in I\}$, denoted $\prod_{i\in I}^{W} G_i$,

Weak Direct Products

Definition (8.3)**.** Let $\{G_i \mid i \in I\}$ be a family of groups.

- The (external) weak direct product of $\{G_i \mid i \in I\}$, denoted $\prod_{i \in I}^{W} G_i$, is the set of all $(a_i)_{i \in I} \in \prod_{i \in I} G_i$ such that $a_i = e_{G_i}$ for all but finitely many $i \in I$.

Weak Direct Products

Definition (8.3)**.** Let $\{G_i \mid i \in I\}$ be a family of groups.

- The (external) weak direct product of $\{G_i \mid i \in I\}$, denoted $\prod_{i \in I}^{W} G_i$, is the set of all $(a_i)_{i \in I} \in \prod_{i \in I} G_i$ such that $a_i = e_{G_i}$ for all but finitely many $i \in I$.

Weak Direct Products

Definition (8.3)**.** Let $\{G_i \mid i \in I\}$ be a family of groups.

- The (external) weak direct product of $\{G_i \mid i \in I\}$, denoted $\prod_{i\in I}^{W} G_i$, is the set of all $(a_i)_{i\in I} \in \prod_{i\in I} G_i$ such that $a_i = e_{G_i}$ for all but finitely many $i \in I$.
- If all the groups G_i are (additive) abelian,

Weak Direct Products

Definition (8.3)**.** Let $\{G_i \mid i \in I\}$ be a family of groups.

- The (external) weak direct product of $\{G_i \mid i \in I\}$, denoted $\prod_{i\in I}^{W} G_i$, is the set of all $(a_i)_{i\in I} \in \prod_{i\in I} G_i$ such that $a_i = e_{G_i}$ for all but finitely many $i \in I$.
- If all the groups G_i are (additive) abelian, $\prod_{i\in I}^{W} G_i$ is usually called the (external) direct sum

Weak Direct Products

Definition (8.3). Let $\{G_i \mid i \in I\}$ be a family of groups.

- The (external) weak direct product of $\{G_i \mid i \in I\}$, denoted $\prod_{i\in I}^{W} G_i$, is the set of all $(a_i)_{i\in I} \in \prod_{i\in I} G_i$ such that $a_i = e_{G_i}$ for all but finitely many $i \in I$.
- If all the groups G_i are (additive) abelian, $\prod_{i\in I}^{W} G_i$ is usually called the (external) direct sum and is denoted $\sum_{i\in I} G_i$

Weak Direct Products

Definition (8.3)**.** Let $\{G_i \mid i \in I\}$ be a family of groups.

- The (external) weak direct product of $\{G_i \mid i \in I\}$, denoted $\prod_{i\in I}^{W} G_i$, is the set of all $(a_i)_{i\in I} \in \prod_{i\in I} G_i$ such that $a_i = e_{G_i}$ for all but finitely many $i \in I$.
- If all the groups G_i are (additive) abelian, $\prod_{i\in I}^{W} G_i$ is usually called the (external) direct sum and is denoted $\sum_{i\in I} G_i$ (or $\bigoplus_{i\in I} G_i$).

Weak Direct Products

Definition (8.3)**.** Let $\{G_i \mid i \in I\}$ be a family of groups.

- The (external) weak direct product of $\{G_i \mid i \in I\}$, denoted $\prod_{i\in I}^{W} G_i$, is the set of all $(a_i)_{i\in I} \in \prod_{i\in I} G_i$ such that $a_i = e_{G_i}$ for all but finitely many $i \in I$.
- If all the groups G_i are (additive) abelian, $\prod_{i\in I}^{W} G_i$ is usually called the (external) direct sum and is denoted $\sum_{i\in I} G_i$ (or $\bigoplus_{i\in I} G_i$).

Remark. If I is finite,

Weak Direct Products

Definition (8.3). Let $\{G_i \mid i \in I\}$ be a family of groups.

- The (external) weak direct product of $\{G_i \mid i \in I\}$, denoted $\prod_{i\in I}^{W} G_i$, is the set of all $(a_i)_{i\in I} \in \prod_{i\in I} G_i$ such that $a_i = e_{G_i}$ for all but finitely many $i \in I$.
- If all the groups G_i are (additive) abelian, $\prod_{i\in I}^{W} G_i$ is usually called the (external) direct sum and is denoted $\sum_{i\in I} G_i$ (or $\bigoplus_{i\in I} G_i$).

Remark. If I is finite, the weak direct product and the direct product coincide.

Weak Direct Products

Definition (8.3)**.** Let $\{G_i \mid i \in I\}$ be a family of groups.

- The (external) weak direct product of $\{G_i \mid i \in I\}$, denoted $\prod_{i\in I}^{W} G_i$, is the set of all $(a_i)_{i\in I} \in \prod_{i\in I} G_i$ such that $a_i = e_{G_i}$ for all but finitely many $i \in I$.
- If all the groups G_i are (additive) abelian, $\prod_{i\in I}^{W} G_i$ is usually called the (external) direct sum and is denoted $\sum_{i\in I} G_i$ (or $\bigoplus_{i\in I} G_i$).

Remark. If I is finite, the weak direct product and the direct product coincide.

Next, we will show that $\prod_{i\in I}^{W} G_i$ is a normal subgroup of $\prod_{i\in I} G_i$

Weak Direct Products

Definition (8.3)**.** Let $\{G_i \mid i \in I\}$ be a family of groups.

- The (external) weak direct product of $\{G_i \mid i \in I\}$, denoted $\prod_{i\in I}^{W} G_i$, is the set of all $(a_i)_{i\in I} \in \prod_{i\in I} G_i$ such that $a_i = e_{G_i}$ for all but finitely many $i \in I$.
- If all the groups G_i are (additive) abelian, $\prod_{i\in I}^{W} G_i$ is usually called the (external) direct sum and is denoted $\sum_{i\in I} G_i$ (or $\bigoplus_{i\in I} G_i$).

Remark. If I is finite, the weak direct product and the direct product coincide.

Next, we will show that $\prod_{i\in I}^{W} G_i$ is a normal subgroup of $\prod_{i\in I} G_i$ and construct, for each $k \in I$,

Weak Direct Products

Definition (8.3). Let $\{G_i \mid i \in I\}$ be a family of groups.

- The (external) weak direct product of $\{G_i \mid i \in I\}$, denoted $\prod_{i \in I}^{W} G_i$, is the set of all $(a_i)_{i \in I} \in \prod_{i \in I} G_i$ such that $a_i = e_{G_i}$ for all but finitely many $i \in I$.
- If all the groups G_i are (additive) abelian, $\prod_{i \in I}^{W} G_i$ is usually called the (external) direct sum and is denoted $\sum_{i \in I} G_i$ (or $\bigoplus_{i \in I} G_i$).

Remark. If I is finite, the weak direct product and the direct product coincide.

Next, we will show that $\prod_{i \in I}^{W} G_i$ is a normal subgroup of $\prod_{i \in I} G_i$ and construct, for each $k \in I$, the canonical injection $\iota_k : G_k \to \prod_{i \in I}^{W} G_i$.

Theorem (8.4)

Let $\{G_i \mid i \in I\}$ be a family of groups.

Theorem (8.4)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- $\prod_{i\in I}^{W} G_i$ is a normal subgroup of $\prod_{i\in I} G_i$.

Theorem (8.4)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- $\prod_{i\in I}^{W} G_i$ is a normal subgroup of $\prod_{i\in I} G_i$.

Proof. First note that $e_{\prod_{i\in I} G_i}$

Theorem (8.4)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- $\prod_{i \in I}^{W} G_i$ is a normal subgroup of $\prod_{i \in I} G_i$.

Proof. First note that $e_{\prod_{i \in I} G_i} = (e_{G_i})_{i \in I}$

Theorem (8.4)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- $\prod_{i\in I}^{W} G_i$ is a normal subgroup of $\prod_{i\in I} G_i$.

Proof. First note that $e_{\prod_{i\in I} G_i} = (e_{G_i})_{i\in I} \in \prod_{i\in I}^{W} G_i$.

Theorem (8.4)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- $\prod_{i\in I}^{W} G_i$ is a normal subgroup of $\prod_{i\in I} G_i$.

Proof. First note that $e_{\prod_{i\in I} G_i} = (e_{G_i})_{i\in I} \in \prod_{i\in I}^{W} G_i$.
Secondly, note that $\left((a_i)_{i\in I}\right)^{-1}$

Theorem (8.4)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- $\prod_{i\in I}^{W} G_i$ is a normal subgroup of $\prod_{i\in I} G_i$.

Proof. First note that $e_{\prod_{i\in I} G_i} = (e_{G_i})_{i\in I} \in \prod_{i\in I}^{W} G_i$.

Secondly, note that $\left((a_i)_{i\in I}\right)^{-1} = (a_i^{-1})_{i\in I}$.

Theorem (8.4)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- $\prod_{i\in I}^{W} G_i$ is a normal subgroup of $\prod_{i\in I} G_i$.

Proof. First note that $e_{\prod_{i\in I} G_i} = (e_{G_i})_{i\in I} \in \prod_{i\in I}^{W} G_i$. Secondly, note that $\left((a_i)_{i\in I}\right)^{-1} = (a_i^{-1})_{i\in I}$. Hence, if $(a_i)_{i\in I} \in \prod_{i\in I}^{W} G_i$,

Theorem (8.4)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- $\prod_{i\in I}^{W} G_i$ is a normal subgroup of $\prod_{i\in I} G_i$.

Proof. First note that $e_{\prod_{i\in I} G_i} = (e_{G_i})_{i\in I} \in \prod_{i\in I}^{W} G_i$.
Secondly, note that $\big((a_i)_{i\in I}\big)^{-1} = (a_i^{-1})_{i\in I}$. Hence, if $(a_i)_{i\in I} \in \prod_{i\in I}^{W} G_i$, i.e., $a_i = e_{G_i}$ for all but finitely many $i \in I$,

Theorem (8.4)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- $\prod_{i\in I}^{W} G_i$ is a normal subgroup of $\prod_{i\in I} G_i$.

Proof. First note that $e_{\prod_{i\in I} G_i} = (e_{G_i})_{i\in I} \in \prod_{i\in I}^{W} G_i$.
Secondly, note that $\big((a_i)_{i\in I}\big)^{-1} = (a_i^{-1})_{i\in I}$. Hence, if
$(a_i)_{i\in I} \in \prod_{i\in I}^{W} G_i$, i.e., $a_i = e_{G_i}$ for all but finitely many $i \in I$,
then $a_i^{-1} = e_{G_i}$ for all but finitely many $i \in I$,

Theorem (8.4)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- $\prod_{i\in I}^{W} G_i$ is a normal subgroup of $\prod_{i\in I} G_i$.

Proof. First note that $e_{\prod_{i\in I} G_i} = (e_{G_i})_{i\in I} \in \prod_{i\in I}^{W} G_i$.
Secondly, note that $\left((a_i)_{i\in I}\right)^{-1} = (a_i^{-1})_{i\in I}$. Hence, if $(a_i)_{i\in I} \in \prod_{i\in I}^{W} G_i$, i.e., $a_i = e_{G_i}$ for all but finitely many $i \in I$, then $a_i^{-1} = e_{G_i}$ for all but finitely many $i \in I$, i.e., $\left((a_i)_{i\in I}\right)^{-1} \in \prod_{i\in I}^{W} G_i$.

Theorem (8.4)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- $\prod_{i\in I}^{W} G_i$ is a normal subgroup of $\prod_{i\in I} G_i$.

Proof. First note that $e_{\prod_{i\in I} G_i} = (e_{G_i})_{i\in I} \in \prod_{i\in I}^{W} G_i$.
Secondly, note that $\left((a_i)_{i\in I}\right)^{-1} = (a_i^{-1})_{i\in I}$. Hence, if $(a_i)_{i\in I} \in \prod_{i\in I}^{W} G_i$, i.e., $a_i = e_{G_i}$ for all but finitely many $i \in I$, then $a_i^{-1} = e_{G_i}$ for all but finitely many $i \in I$, i.e., $\left((a_i)_{i\in I}\right)^{-1} \in \prod_{i\in I}^{W} G_i$.
Thirdly, if $(a_i)_{i\in I}, (b_i)_{i\in I} \in \prod_{i\in I}^{W} G_i$,

Theorem (8.4)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- $\prod_{i\in I}^W G_i$ is a normal subgroup of $\prod_{i\in I} G_i$.

Proof. First note that $e_{\prod_{i\in I} G_i} = (e_{G_i})_{i\in I} \in \prod_{i\in I}^W G_i$.
Secondly, note that $\left((a_i)_{i\in I}\right)^{-1} = (a_i^{-1})_{i\in I}$. Hence, if $(a_i)_{i\in I} \in \prod_{i\in I}^W G_i$, i.e., $a_i = e_{G_i}$ for all but finitely many $i \in I$, then $a_i^{-1} = e_{G_i}$ for all but finitely many $i \in I$, i.e., $\left((a_i)_{i\in I}\right)^{-1} \in \prod_{i\in I}^W G_i$.
Thirdly, if $(a_i)_{i\in I}, (b_i)_{i\in I} \in \prod_{i\in I}^W G_i$, there are only finitely many $i \in I$ such that $a_i \neq e_{G_i}$ or $b_i \neq e_{G_i}$,

Theorem (8.4)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- $\prod_{i\in I}^{W} G_i$ is a normal subgroup of $\prod_{i\in I} G_i$.

Proof. First note that $e_{\prod_{i\in I} G_i} = (e_{G_i})_{i\in I} \in \prod_{i\in I}^{W} G_i$.
Secondly, note that $\big((a_i)_{i\in I}\big)^{-1} = (a_i^{-1})_{i\in I}$. Hence, if $(a_i)_{i\in I} \in \prod_{i\in I}^{W} G_i$, i.e., $a_i = e_{G_i}$ for all but finitely many $i \in I$, then $a_i^{-1} = e_{G_i}$ for all but finitely many $i \in I$, i.e., $\big((a_i)_{i\in I}\big)^{-1} \in \prod_{i\in I}^{W} G_i$.
Thirdly, if $(a_i)_{i\in I}, (b_i)_{i\in I} \in \prod_{i\in I}^{W} G_i$, there are only finitely many $i \in I$ such that $a_i \neq e_{G_i}$ or $b_i \neq e_{G_i}$, whence there are only finitely many $i \in I$ such that $a_i b_i \neq e_{G_i}$,

Theorem (8.4)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- $\prod_{i\in I}^{W} G_i$ is a normal subgroup of $\prod_{i\in I} G_i$.

Proof. First note that $e_{\prod_{i\in I} G_i} = (e_{G_i})_{i\in I} \in \prod_{i\in I}^{W} G_i$.
Secondly, note that $\left((a_i)_{i\in I}\right)^{-1} = (a_i^{-1})_{i\in I}$. Hence, if $(a_i)_{i\in I} \in \prod_{i\in I}^{W} G_i$, i.e., $a_i = e_{G_i}$ for all but finitely many $i \in I$, then $a_i^{-1} = e_{G_i}$ for all but finitely many $i \in I$, i.e., $\left((a_i)_{i\in I}\right)^{-1} \in \prod_{i\in I}^{W} G_i$.
Thirdly, if $(a_i)_{i\in I}, (b_i)_{i\in I} \in \prod_{i\in I}^{W} G_i$, there are only finitely many $i \in I$ such that $a_i \neq e_{G_i}$ or $b_i \neq e_{G_i}$, whence there are only finitely many $i \in I$ such that $a_i b_i \neq e_{G_i}$, and so $(a_i)_{i\in I}(b_i)_{i\in I} = (a_i b_i)_{i\in I} \in \prod_{i\in I}^{W} G_i$.

Theorem (8.4)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- $\prod_{i\in I}^{W} G_i$ is a normal subgroup of $\prod_{i\in I} G_i$.

Proof. First note that $e_{\prod_{i\in I} G_i} = (e_{G_i})_{i\in I} \in \prod_{i\in I}^{W} G_i$. (Identity)
Secondly, note that $\left((a_i)_{i\in I}\right)^{-1} = (a_i^{-1})_{i\in I}$. Hence, if $(a_i)_{i\in I} \in \prod_{i\in I}^{W} G_i$, i.e., $a_i = e_{G_i}$ for all but finitely many $i \in I$, then $a_i^{-1} = e_{G_i}$ for all but finitely many $i \in I$, i.e., $\left((a_i)_{i\in I}\right)^{-1} \in \prod_{i\in I}^{W} G_i$. (Inverse)
Thirdly, if $(a_i)_{i\in I}, (b_i)_{i\in I} \in \prod_{i\in I}^{W} G_i$, there are only finitely many $i \in I$ such that $a_i \neq e_{G_i}$ or $b_i \neq e_{G_i}$, whence there are only finitely many $i \in I$ such that $a_i b_i \neq e_{G_i}$, and so $(a_i)_{i\in I}(b_i)_{i\in I} = (a_i b_i)_{i\in I} \in \prod_{i\in I}^{W} G_i$. (Closed)
Therefore, $\prod_{i\in I}^{W} G_i$ is a subgroup of $\prod_{i\in I} G_i$.

Theorem (8.4)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- $\prod_{i\in I}^{W} G_i$ is a normal subgroup of $\prod_{i\in I} G_i$.

Proof. Finally, if $(a_i)_{i\in I} \in \prod_{i\in I}^{W} G_i$ and if $(b_i)_{i\in I} \in \prod_{i\in I} G_i$,

Theorem (8.4)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- $\prod_{i\in I}^{W} G_i$ is a normal subgroup of $\prod_{i\in I} G_i$.

Proof. Finally, if $(a_i)_{i\in I} \in \prod_{i\in I}^{W} G_i$ and if $(b_i)_{i\in I} \in \prod_{i\in I} G_i$,
$\left((b_i)_{i\in I}\right)^{-1}(a_i)_{i\in I}(b_i)_{i\in I}$

Theorem (8.4)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- $\prod_{i\in I}^{W} G_i$ is a normal subgroup of $\prod_{i\in I} G_i$.

Proof. Finally, if $(a_i)_{i\in I} \in \prod_{i\in I}^{W} G_i$ and if $(b_i)_{i\in I} \in \prod_{i\in I} G_i$,

$$\big((b_i)_{i\in I}\big)^{-1}(a_i)_{i\in I}(b_i)_{i\in I} = (b_i^{-1}a_ib_i)_{i\in I}$$

Theorem (8.4)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- $\prod_{i\in I}^{W} G_i$ is a normal subgroup of $\prod_{i\in I} G_i$.

Proof. Finally, if $(a_i)_{i\in I} \in \prod_{i\in I}^{W} G_i$ and if $(b_i)_{i\in I} \in \prod_{i\in I} G_i$,

$$\left((b_i)_{i\in I}\right)^{-1}(a_i)_{i\in I}(b_i)_{i\in I} = (b_i^{-1}a_ib_i)_{i\in I}$$

with $b_i^{-1}a_ib_i$

Theorem (8.4)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- $\prod_{i\in I}^{W} G_i$ is a normal subgroup of $\prod_{i\in I} G_i$.

Proof. Finally, if $(a_i)_{i\in I} \in \prod_{i\in I}^{W} G_i$ and if $(b_i)_{i\in I} \in \prod_{i\in I} G_i$,

$$\left((b_i)_{i\in I}\right)^{-1}(a_i)_{i\in I}(b_i)_{i\in I} = (b_i^{-1}a_ib_i)_{i\in I}$$

with $b_i^{-1}a_ib_i = b_i^{-1}e_{G_i}b_i$ for all but finitely many $i \in I$,

Theorem (8.4)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- $\prod_{i\in I}^{W} G_i$ is a normal subgroup of $\prod_{i\in I} G_i$.

Proof. Finally, if $(a_i)_{i\in I} \in \prod_{i\in I}^{W} G_i$ and if $(b_i)_{i\in I} \in \prod_{i\in I} G_i$,

$$\left((b_i)_{i\in I}\right)^{-1}(a_i)_{i\in I}(b_i)_{i\in I} = (b_i^{-1}a_ib_i)_{i\in I}$$

with $b_i^{-1}a_ib_i = b_i^{-1}e_{G_i}b_i = e_{G_i}$ for all but finitely many $i \in I$,

Theorem (8.4)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- $\prod_{i\in I}^{W} G_i$ is a normal subgroup of $\prod_{i\in I} G_i$.

Proof. Finally, if $(a_i)_{i\in I} \in \prod_{i\in I}^{W} G_i$ and if $(b_i)_{i\in I} \in \prod_{i\in I} G_i$,

$$\big((b_i)_{i\in I}\big)^{-1}(a_i)_{i\in I}(b_i)_{i\in I} = (b_i^{-1}a_ib_i)_{i\in I}$$

with $b_i^{-1}a_ib_i = b_i^{-1}e_{G_i}b_i = e_{G_i}$ for all but finitely many $i \in I$, whence $\big((b_i)_{i\in I}\big)^{-1}(a_i)_{i\in I}(b_i)_{i\in I} \in \prod_{i\in I}^{W} G_i$.

Theorem (8.4)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- $\prod_{i\in I}^{W} G_i$ is a normal subgroup of $\prod_{i\in I} G_i$.

Proof. Finally, if $(a_i)_{i\in I} \in \prod_{i\in I}^{W} G_i$ and if $(b_i)_{i\in I} \in \prod_{i\in I} G_i$,

$$\big((b_i)_{i\in I}\big)^{-1}(a_i)_{i\in I}(b_i)_{i\in I} = (b_i^{-1}a_ib_i)_{i\in I}$$

with $b_i^{-1}a_ib_i = b_i^{-1}e_{G_i}b_i = e_{G_i}$ for all but finitely many $i \in I$, whence $\big((b_i)_{i\in I}\big)^{-1}(a_i)_{i\in I}(b_i)_{i\in I} \in \prod_{i\in I}^{W} G_i$. This shows that $\prod_{i\in I}^{W} G_i$ is a normal subgroup of $\prod_{i\in I} G_i$.

Theorem (8.4)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- $\prod_{i \in I}^{W} G_i$ is a normal subgroup of $\prod_{i \in I} G_i$.
- For each $k \in I$,

Theorem (8.4)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- $\prod_{i\in I}^{W} G_i$ is a normal subgroup of $\prod_{i\in I} G_i$.
- For each $k \in I$, the map $\iota_k : G_k \to \prod_{i\in I}^{W} G_i$

Theorem (8.4)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- $\prod_{i\in I}^{W} G_i$ is a normal subgroup of $\prod_{i\in I} G_i$.
- For each $k \in I$, the map $\iota_k : G_k \to \prod_{i\in I}^{W} G_i$ given by

$$\iota_k(a) = (a_i)_{i\in I},$$

Theorem (8.4)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- $\prod_{i\in I}^{W} G_i$ is a normal subgroup of $\prod_{i\in I} G_i$.
- For each $k \in I$, the map $\iota_k : G_k \to \prod_{i\in I}^{W} G_i$ given by

$$\iota_k(a) = (a_i)_{i\in I}, \quad \text{where} \begin{cases} a_k = a \\ a_i = e_{G_i} \ \forall\, i \neq k \end{cases}$$

Theorem (8.4)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- $\prod_{i\in I}^{W} G_i$ is a normal subgroup of $\prod_{i\in I} G_i$.
- For each $k \in I$, the map $\iota_k : G_k \to \prod_{i\in I}^{W} G_i$ given by

$$\iota_k(a) = (a_i)_{i\in I}, \quad \text{where} \begin{cases} a_k = a \\ a_i = e_{G_i} \ \forall\, i \neq k \end{cases}$$

  is a monomorphism of groups.

Theorem (8.4)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- $\prod_{i\in I}^{W} G_i$ is a normal subgroup of $\prod_{i\in I} G_i$.
- For each $k \in I$, the map $\iota_k : G_k \to \prod_{i\in I}^{W} G_i$ given by

$$\iota_k(a) = (a_i)_{i\in I}, \quad \text{where} \begin{cases} a_k = a \\ a_i = e_{G_i} \ \forall\, i \neq k \end{cases}$$

  is a monomorphism of groups.

Proof. For $a, b \in G_k$,

Theorem (8.4)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- $\prod_{i\in I}^{W} G_i$ is a normal subgroup of $\prod_{i\in I} G_i$.
- For each $k \in I$, the map $\iota_k : G_k \to \prod_{i\in I}^{W} G_i$ given by

$$\iota_k(a) = (a_i)_{i\in I}, \quad \text{where} \begin{cases} a_k = a \\ a_i = e_{G_i} \ \forall\, i \neq k \end{cases}$$

  is a monomorphism of groups.

Proof. For $a, b \in G_k$, let $(a_i)_{i\in I}$, $(b_i)_{i\in I}$, $(c_i)_{i\in I} \in \prod_{i\in I}^{W} G_i$

Theorem (8.4)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- $\prod_{i\in I}^{W} G_i$ is a normal subgroup of $\prod_{i\in I} G_i$.
- For each $k \in I$, the map $\iota_k : G_k \to \prod_{i\in I}^{W} G_i$ given by

$$\iota_k(a) = (a_i)_{i\in I}, \qquad \text{where} \begin{cases} a_k = a \\ a_i = e_{G_i}\ \forall\, i \neq k \end{cases}$$

  is a monomorphism of groups.

Proof. For $a, b \in G_k$, let $(a_i)_{i\in I}$, $(b_i)_{i\in I}$, $(c_i)_{i\in I} \in \prod_{i\in I}^{W} G_i$ such that $\begin{cases} a_k = a,\ b_k = b,\ \text{and } c_k = ab, \\ a_i = b_i = c_i = e_{G_i}\ \forall i \neq k. \end{cases}$

Theorem (8.4)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- $\prod_{i\in I}^{W} G_i$ is a normal subgroup of $\prod_{i\in I} G_i$.
- For each $k \in I$, the map $\iota_k : G_k \to \prod_{i\in I}^{W} G_i$ given by

$$\iota_k(a) = (a_i)_{i\in I}, \quad \text{where} \begin{cases} a_k = a \\ a_i = e_{G_i} \ \forall\, i \neq k \end{cases}$$

is a monomorphism of groups.

Proof. For $a, b \in G_k$, let $(a_i)_{i\in I}$, $(b_i)_{i\in I}$, $(c_i)_{i\in I} \in \prod_{i\in I}^{W} G_i$ such that $\begin{cases} a_k = a,\ b_k = b,\ \text{and } c_k = ab, \\ a_i = b_i = c_i = e_{G_i}\ \forall i \neq k. \end{cases}$ Then $\iota_k(a) = (a_i)_{i\in I}$,

Theorem (8.4)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- $\prod^W_{i\in I} G_i$ is a normal subgroup of $\prod_{i\in I} G_i$.
- For each $k \in I$, the map $\iota_k : G_k \to \prod^W_{i\in I} G_i$ given by

$$\iota_k(a) = (a_i)_{i\in I}, \quad \text{where} \begin{cases} a_k = a \\ a_i = e_{G_i}\ \forall\ i \neq k \end{cases}$$

  is a monomorphism of groups.

Proof. For $a, b \in G_k$, let $(a_i)_{i\in I}$, $(b_i)_{i\in I}$, $(c_i)_{i\in I} \in \prod^W_{i\in I} G_i$ such that $\begin{cases} a_k = a,\ b_k = b,\ \text{and } c_k = ab, \\ a_i = b_i = c_i = e_{G_i}\ \forall i \neq k. \end{cases}$ Then $\iota_k(a) = (a_i)_{i\in I}$, $\iota_k(b) = (b_i)_{i\in I}$,

Theorem (8.4)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- $\prod_{i\in I}^{W} G_i$ is a normal subgroup of $\prod_{i\in I} G_i$.
- For each $k \in I$, the map $\iota_k : G_k \to \prod_{i\in I}^{W} G_i$ given by

$$\iota_k(a) = (a_i)_{i\in I}, \quad \text{where} \begin{cases} a_k = a \\ a_i = e_{G_i} \ \forall\, i \neq k \end{cases}$$

  is a monomorphism of groups.

Proof. For $a, b \in G_k$, let $(a_i)_{i\in I}$, $(b_i)_{i\in I}$, $(c_i)_{i\in I} \in \prod_{i\in I}^{W} G_i$ such that $\begin{cases} a_k = a,\ b_k = b,\ \text{and } c_k = ab, \\ a_i = b_i = c_i = e_{G_i}\ \forall i \neq k. \end{cases}$ Then $\iota_k(a) = (a_i)_{i\in I}$, $\iota_k(b) = (b_i)_{i\in I}$, and $\iota_k(ab) = (c_i)_{i\in I}$.

Theorem (8.4)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- $\prod_{i \in I}^{W} G_i$ is a normal subgroup of $\prod_{i \in I} G_i$.
- For each $k \in I$, the map $\iota_k : G_k \to \prod_{i \in I}^{W} G_i$ given by

$$\iota_k(a) = (a_i)_{i \in I}, \quad \text{where} \begin{cases} a_k = a \\ a_i = e_{G_i} \ \forall \ i \neq k \end{cases}$$

  is a monomorphism of groups.

Proof. For $a, b \in G_k$, let $(a_i)_{i \in I}$, $(b_i)_{i \in I}$, $(c_i)_{i \in I} \in \prod_{i \in I}^{W} G_i$ such that $\begin{cases} a_k = a, \ b_k = b, \ \text{and } c_k = ab, \\ a_i = b_i = c_i = e_{G_i} \ \forall i \neq k. \end{cases}$ Then $\iota_k(a) = (a_i)_{i \in I}$, $\iota_k(b) = (b_i)_{i \in I}$, and $\iota_k(ab) = (c_i)_{i \in I}$. Since $(a_i)_{i \in I}(b_i)_{i \in I} = (a_i b_i)_{i \in I}$

Theorem (8.4)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- $\prod_{i \in I}^{W} G_i$ is a normal subgroup of $\prod_{i \in I} G_i$.
- For each $k \in I$, the map $\iota_k : G_k \to \prod_{i \in I}^{W} G_i$ given by

$$\iota_k(a) = (a_i)_{i \in I}, \quad \text{where} \begin{cases} a_k = a \\ a_i = e_{G_i} \ \forall\, i \neq k \end{cases}$$

is a monomorphism of groups.

Proof. For $a, b \in G_k$, let $(a_i)_{i \in I}$, $(b_i)_{i \in I}$, $(c_i)_{i \in I} \in \prod_{i \in I}^{W} G_i$ such that $\begin{cases} a_k = a,\ b_k = b,\ \text{and } c_k = ab, \\ a_i = b_i = c_i = e_{G_i}\ \forall i \neq k. \end{cases}$ Then $\iota_k(a) = (a_i)_{i \in I}$, $\iota_k(b) = (b_i)_{i \in I}$, and $\iota_k(ab) = (c_i)_{i \in I}$. Since $(a_i)_{i \in I}(b_i)_{i \in I} = (a_i b_i)_{i \in I} = (c_i)_{i \in I}$,

Theorem (8.4)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- $\prod_{i\in I}^{W} G_i$ is a normal subgroup of $\prod_{i\in I} G_i$.
- For each $k \in I$, the map $\iota_k : G_k \to \prod_{i\in I}^{W} G_i$ given by

$$\iota_k(a) = (a_i)_{i\in I}, \quad \text{where} \begin{cases} a_k = a \\ a_i = e_{G_i} \ \forall\, i \neq k \end{cases}$$

  is a monomorphism of groups.

Proof. For $a, b \in G_k$, let $(a_i)_{i\in I}$, $(b_i)_{i\in I}$, $(c_i)_{i\in I} \in \prod_{i\in I}^{W} G_i$ such that $\begin{cases} a_k = a,\ b_k = b,\ \text{and } c_k = ab, \\ a_i = b_i = c_i = e_{G_i}\ \forall i \neq k. \end{cases}$ Then $\iota_k(a) = (a_i)_{i\in I}$, $\iota_k(b) = (b_i)_{i\in I}$, and $\iota_k(ab) = (c_i)_{i\in I}$. Since $(a_i)_{i\in I}(b_i)_{i\in I} = (a_ib_i)_{i\in I} = (c_i)_{i\in I}$, $\iota_k(a)\iota_k(b) = \iota_k(ab)$.

Theorem (8.4)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- $\prod_{i\in I}^{W} G_i$ is a normal subgroup of $\prod_{i\in I} G_i$.
- For each $k \in I$, the map $\iota_k : G_k \to \prod_{i\in I}^{W} G_i$ given by

$$\iota_k(a) = (a_i)_{i\in I}, \quad \text{where} \begin{cases} a_k = a \\ a_i = e_{G_i}\ \forall\, i \neq k \end{cases}$$

  is a monomorphism of groups.

Proof. For $a, b \in G_k$, let $(a_i)_{i\in I}$, $(b_i)_{i\in I}$, $(c_i)_{i\in I} \in \prod_{i\in I}^{W} G_i$ such that $\begin{cases} a_k = a,\ b_k = b,\ \text{and } c_k = ab, \\ a_i = b_i = c_i = e_{G_i}\ \forall i \neq k. \end{cases}$ Then $\iota_k(a) = (a_i)_{i\in I}$, $\iota_k(b) = (b_i)_{i\in I}$, and $\iota_k(ab) = (c_i)_{i\in I}$. Since $(a_i)_{i\in I}(b_i)_{i\in I} = (a_ib_i)_{i\in I} = (c_i)_{i\in I}$, $\iota_k(a)\iota_k(b) = \iota_k(ab)$. Hence ι_k is a group homomorphism.

Theorem (8.4)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- $\prod_{i\in I}^{W} G_i$ is a normal subgroup of $\prod_{i\in I} G_i$.
- For each $k \in I$, the map $\iota_k : G_k \to \prod_{i\in I}^{W} G_i$ given by

$$\iota_k(a) = (a_i)_{i\in I}, \quad \text{where} \begin{cases} a_k = a \\ a_i = e_{G_i} \ \forall\, i \neq k \end{cases}$$

  is a monomorphism of groups.

Proof. Moreover, $a \in \operatorname{Ker} \iota_k$

Theorem (8.4)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- $\prod_{i\in I}^{W} G_i$ is a normal subgroup of $\prod_{i\in I} G_i$.
- For each $k \in I$, the map $\iota_k : G_k \to \prod_{i\in I}^{W} G_i$ given by

$$\iota_k(a) = (a_i)_{i\in I}, \quad \text{where} \begin{cases} a_k = a \\ a_i = e_{G_i} \ \forall\, i \neq k \end{cases}$$

  is a monomorphism of groups.

Proof. Moreover, $a \in \operatorname{Ker} \iota_k \iff \iota_k(a) = (e_{G_i})_{i\in I}$

Theorem (8.4)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- $\prod_{i\in I}^{W} G_i$ is a normal subgroup of $\prod_{i\in I} G_i$.
- For each $k \in I$, the map $\iota_k : G_k \to \prod_{i\in I}^{W} G_i$ given by

$$\iota_k(a) = (a_i)_{i\in I}, \quad \text{where} \begin{cases} a_k = a \\ a_i = e_{G_i} \ \forall\, i \neq k \end{cases}$$

  is a monomorphism of groups.

Proof. Moreover, $a \in \operatorname{Ker} \iota_k \iff \iota_k(a) = (e_{G_i})_{i\in I}$ $\iff a = e_{G_k}$,

Theorem (8.4)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- $\prod_{i\in I}^{W} G_i$ is a normal subgroup of $\prod_{i\in I} G_i$.
- For each $k \in I$, the map $\iota_k : G_k \to \prod_{i\in I}^{W} G_i$ given by

$$\iota_k(a) = (a_i)_{i\in I}, \quad \text{where} \begin{cases} a_k = a \\ a_i = e_{G_i} \ \forall\, i \neq k \end{cases}$$

  is a monomorphism of groups.

Proof. Moreover, $a \in \operatorname{Ker} \iota_k \iff \iota_k(a) = (e_{G_i})_{i\in I}$ $\iff a = e_{G_k}$, so ι_k is a monomorphism.

Theorem (8.4)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- $\prod_{i\in I}^{W} G_i$ is a normal subgroup of $\prod_{i\in I} G_i$.
- For each $k \in I$, the map $\iota_k : G_k \to \prod_{i\in I}^{W} G_i$ given by

$$\iota_k(a) = (a_i)_{i\in I}, \quad \text{where} \begin{cases} a_k = a \\ a_i = e_{G_i} \ \forall\, i \neq k \end{cases}$$

  is a monomorphism of groups.
- For each $k \in I$,

Theorem (8.4)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- $\prod_{i\in I}^{W} G_i$ is a normal subgroup of $\prod_{i\in I} G_i$.
- For each $k \in I$, the map $\iota_k : G_k \to \prod_{i\in I}^{W} G_i$ given by

$$\iota_k(a) = (a_i)_{i\in I}, \quad \text{where} \begin{cases} a_k = a \\ a_i = e_{G_i} \ \forall\, i \neq k \end{cases}$$

  is a monomorphism of groups.
- For each $k \in I$, $\iota_k(G_k)$ is a normal subgroup of $\prod_{i\in I} G_i$.

Theorem (8.4)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- $\prod_{i\in I}^{W} G_i$ is a normal subgroup of $\prod_{i\in I} G_i$.
- For each $k \in I$, the map $\iota_k : G_k \to \prod_{i\in I}^{W} G_i$ given by

$$\iota_k(a) = (a_i)_{i\in I}, \quad \text{where} \begin{cases} a_k = a \\ a_i = e_{G_i} \ \forall\, i \neq k \end{cases}$$

  is a monomorphism of groups.
- For each $k \in I$, $\iota_k(G_k)$ is a normal subgroup of $\prod_{i\in I} G_i$.

Proof. $\iota_k(G_k)$

Theorem (8.4)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- $\prod_{i \in I}^{W} G_i$ is a normal subgroup of $\prod_{i \in I} G_i$.
- For each $k \in I$, the map $\iota_k : G_k \to \prod_{i \in I}^{W} G_i$ given by

$$\iota_k(a) = (a_i)_{i \in I}, \quad \text{where} \begin{cases} a_k = a \\ a_i = e_{G_i} \ \forall\, i \neq k \end{cases}$$

  is a monomorphism of groups.
- For each $k \in I$, $\iota_k(G_k)$ is a normal subgroup of $\prod_{i \in I} G_i$.

Proof. $\iota_k(G_k) = \{(a_i)_{i \in I} \in \prod_{i \in I} G_i \mid a_i = e_{G_i} \text{ for all } i \neq k\}$.

Theorem (8.4)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- $\prod_{i\in I}^{W} G_i$ is a normal subgroup of $\prod_{i\in I} G_i$.
- For each $k \in I$, the map $\iota_k : G_k \to \prod_{i\in I}^{W} G_i$ given by

$$\iota_k(a) = (a_i)_{i\in I}, \quad \text{where} \begin{cases} a_k = a \\ a_i = e_{G_i} \ \forall\, i \neq k \end{cases}$$

  is a monomorphism of groups.
- For each $k \in I$, $\iota_k(G_k)$ is a normal subgroup of $\prod_{i\in I} G_i$.

Proof. $\iota_k(G_k) = \{(a_i)_{i\in I} \in \prod_{i\in I} G_i \mid a_i = e_{G_i}$ for all $i \neq k\}$.
With a similar proof of $\prod_{i\in I}^{W} G_i$ being a normal subgroup of $\prod_{i\in I} G_i$,

Theorem (8.4)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- $\prod_{i\in I}^{W} G_i$ is a normal subgroup of $\prod_{i\in I} G_i$.
- For each $k \in I$, the map $\iota_k : G_k \to \prod_{i\in I}^{W} G_i$ given by

$$\iota_k(a) = (a_i)_{i\in I}, \quad \text{where} \begin{cases} a_k = a \\ a_i = e_{G_i} \ \forall\, i \neq k \end{cases}$$

  is a monomorphism of groups.
- For each $k \in I$, $\iota_k(G_k)$ is a normal subgroup of $\prod_{i\in I} G_i$.

Proof. $\iota_k(G_k) = \{(a_i)_{i\in I} \in \prod_{i\in I} G_i \mid a_i = e_{G_i}$ for all $i \neq k\}$.
With a similar (but easier) proof of $\prod_{i\in I}^{W} G_i$ being a normal subgroup of $\prod_{i\in I} G_i$,

Theorem (8.4)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- $\prod_{i\in I}^{W} G_i$ is a normal subgroup of $\prod_{i\in I} G_i$.
- For each $k \in I$, the map $\iota_k : G_k \to \prod_{i\in I}^{W} G_i$ given by

$$\iota_k(a) = (a_i)_{i\in I}, \quad \text{where} \begin{cases} a_k = a \\ a_i = e_{G_i} \ \forall\, i \neq k \end{cases}$$

  is a monomorphism of groups.
- For each $k \in I$, $\iota_k(G_k)$ is a normal subgroup of $\prod_{i\in I} G_i$.

Proof. $\iota_k(G_k) = \{(a_i)_{i\in I} \in \prod_{i\in I} G_i \mid a_i = e_{G_i}$ for all $i \neq k\}$.
With a similar (but easier) proof of $\prod_{i\in I}^{W} G_i$ being a normal subgroup of $\prod_{i\in I} G_i$, we can show that $\iota_k(G_k)$ is a normal subgroup of $\prod_{i\in I} G_i$.

Theorem (8.4)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- $\prod_{i\in I}^{W} G_i$ is a normal subgroup of $\prod_{i\in I} G_i$.
- For each $k \in I$, the map $\iota_k : G_k \to \prod_{i\in I}^{W} G_i$ given by

$$\iota_k(a) = (a_i)_{i\in I}, \qquad \text{where } \begin{cases} a_k = a \\ a_i = e_{G_i} \;\forall\, i \neq k \end{cases}$$

  is a monomorphism of groups.
- For each $k \in I$, $\iota_k(G_k)$ is a normal subgroup of $\prod_{i\in I} G_i$.

Definition. The map $\iota_k : G_k \to \prod_{i\in I}^{W} G_i$ are called

Theorem (8.4)

Let $\{G_i \mid i \in I\}$ be a family of groups.

- $\prod_{i\in I}^{W} G_i$ is a normal subgroup of $\prod_{i\in I} G_i$.
- For each $k \in I$, the map $\iota_k : G_k \to \prod_{i\in I}^{W} G_i$ given by

$$\iota_k(a) = (a_i)_{i\in I}, \quad \text{where} \begin{cases} a_k = a \\ a_i = e_{G_i}\ \forall\, i \neq k \end{cases}$$

  is a monomorphism of groups.
- For each $k \in I$, $\iota_k(G_k)$ is a normal subgroup of $\prod_{i\in I} G_i$.

Definition. The map $\iota_k : G_k \to \prod_{i\in I}^{W} G_i$ are called the canonical injections.

Theorem (8.5)

Let $\{A_i \mid i \in I\}$ be a family of abelian groups,

Theorem (8.5)

Let $\{A_i \mid i \in I\}$ be a family of abelian groups, written additively.

Theorem (8.5)

Let $\{A_i \mid i \in I\}$ be a family of abelian groups, written additively.
If B is an abelian group

Theorem (8.5)

Let $\{A_i \mid i \in I\}$ be a family of abelian groups, written additively. If B is an abelian group and if $\{\psi_i : A_i \to B \mid i \in I\}$ is a family of homomorphisms,

Theorem (8.5)

Let $\{A_i \mid i \in I\}$ be a family of abelian groups, written additively. If B is an abelian group and if $\{\psi_i : A_i \to B \mid i \in I\}$ is a family of homomorphisms, then there is a unique homomorphism $\psi : \sum_{i \in I} A_i \to B$

Theorem (8.5)

Let $\{A_i \mid i \in I\}$ be a family of abelian groups, written additively. If B is an abelian group and if $\{\psi_i : A_i \longrightarrow B \mid i \in I\}$ is a family of homomorphisms, then there is a unique homomorphism $\psi : \sum_{i \in I} A_i \longrightarrow B$ such that $\psi \iota_i = \psi_i \; \forall i \in I$;

Theorem (8.5)

Let $\{A_i \mid i \in I\}$ be a family of abelian groups, written additively. If B is an abelian group and if $\{\psi_i : A_i \to B \mid i \in I\}$ is a family of homomorphisms, then there is a unique homomorphism $\psi : \sum_{i \in I} A_i \to B$ such that $\psi \iota_i = \psi_i \; \forall i \in I$; moreover, this property determines $\sum_{i \in I} A_i$ uniquely up to isomorphism.

Theorem (8.5)

Let $\{A_i \mid i \in I\}$ be a family of abelian groups, written additively. If B is an abelian group and if $\{\psi_i : A_i \to B \mid i \in I\}$ is a family of homomorphisms, then there is a unique homomorphism $\psi : \sum_{i \in I} A_i \to B$ such that $\psi \iota_i = \psi_i \; \forall i \in I$; moreover, this property determines $\sum_{i \in I} A_i$ uniquely up to isomorphism. In particular,

Theorem (8.5)

Let $\{A_i \mid i \in I\}$ be a family of abelian groups, written additively. If B is an abelian group and if $\{\psi_i : A_i \to B \mid i \in I\}$ is a family of homomorphisms, then there is a unique homomorphism $\psi : \sum_{i \in I} A_i \to B$ such that $\psi \iota_i = \psi_i \; \forall i \in I$; moreover, this property determines $\sum_{i \in I} A_i$ uniquely up to isomorphism. In particular, $\sum_{i \in I} A_i$ is a coproduct in the category $\mathcal{A}$ of abelian groups.

Theorem (8.5)

Let $\{A_i \mid i \in I\}$ be a family of abelian groups, written additively. If B is an abelian group and if $\{\psi_i : A_i \to B \mid i \in I\}$ is a family of homomorphisms, then there is a unique homomorphism $\psi : \sum_{i \in I} A_i \to B$ such that $\psi \iota_i = \psi_i \; \forall i \in I$; moreover, this property determines $\sum_{i \in I} A_i$ uniquely up to isomorphism. In particular, $\sum_{i \in I} A_i$ is a coproduct in the category $\mathcal{A}$ of abelian groups.

Remark. This theorem is false if the word “abelian” is omitted.

Theorem (8.5)

Let $\{A_i \mid i \in I\}$ be a family of abelian groups, written additively. If B is an abelian group and if $\{\psi_i : A_i \to B \mid i \in I\}$ is a family of homomorphisms, then there is a unique homomorphism $\psi : \sum_{i \in I} A_i \to B$ such that $\psi \iota_i = \psi_i \; \forall i \in I$; moreover, this property determines $\sum_{i \in I} A_i$ uniquely up to isomorphism. In particular, $\sum_{i \in I} A_i$ is a coproduct in the category $\mathcal{A}$ of abelian groups.

Remark. This theorem is false if the word "abelian" is omitted.

Theorem (8.5)

Let $\{A_i \mid i \in I\}$ be a family of abelian groups, written additively. If B is an abelian group and if $\{\psi_i : A_i \to B \mid i \in I\}$ is a family of homomorphisms, then there is a unique homomorphism $\psi : \sum_{i \in I} A_i \to B$ such that $\psi \iota_i = \psi_i \ \forall i \in I$; moreover, this property determines $\sum_{i \in I} A_i$ uniquely up to isomorphism. In particular, $\sum_{i \in I} A_i$ is a coproduct in the category $\mathcal{A}$ of abelian groups.

Remark. This theorem is false if the word "abelian" is omitted.

Theorem (8.5)

Let $\{A_i \mid i \in I\}$ be a family of abelian groups, written additively. If B is an abelian group and if $\{\psi_i : A_i \to B \mid i \in I\}$ is a family of homomorphisms, then there is a unique homomorphism $\psi : \sum_{i \in I} A_i \to B$ such that $\psi \iota_i = \psi_i \ \forall i \in I$; moreover, this property determines $\sum_{i \in I} A_i$ uniquely up to isomorphism. In particular, $\sum_{i \in I} A_i$ is a coproduct in the category $\mathcal{A}$ of abelian groups.

Remark. This theorem is false if the word "abelian" is omitted. The external weak direct product is NOT a coproduct in the category $\mathcal{G}$ of all groups.

Theorem (8.5)

Let $\{A_i \mid i \in I\}$ be a family of abelian groups, written additively. If B is an abelian group and if $\{\psi_i : A_i \to B \mid i \in I\}$ is a family of homomorphisms, then there is a unique homomorphism $\psi : \sum_{i \in I} A_i \to B$ such that $\psi \iota_i = \psi_i \; \forall i \in I$; moreover, this property determines $\sum_{i \in I} A_i$ uniquely up to isomorphism. In particular, $\sum_{i \in I} A_i$ is a coproduct in the category $\mathcal{A}$ of abelian groups.

Remark. This theorem is false if the word "abelian" is omitted. The external weak direct product is NOT a coproduct in the category $\mathcal{G}$ of all groups.

Theorem (8.5)

Let $\{A_i \mid i \in I\}$ be a family of abelian groups, written additively. If B is an abelian group and if $\{\psi_i : A_i \to B \mid i \in I\}$ is a family of homomorphisms, then there is a unique homomorphism $\psi : \sum_{i \in I} A_i \to B$ such that $\psi \iota_i = \psi_i \; \forall i \in I$; moreover, this property determines $\sum_{i \in I} A_i$ uniquely up to isomorphism. In particular, $\sum_{i \in I} A_i$ is a coproduct in the category $\mathcal{A}$ of abelian groups.

Proof. First, note that

Theorem (8.5)

Let $\{A_i \mid i \in I\}$ be a family of abelian groups, written additively. If B is an abelian group and if $\{\psi_i : A_i \to B \mid i \in I\}$ is a family of homomorphisms, then there is a unique homomorphism $\psi : \sum_{i \in I} A_i \to B$ such that $\psi \iota_i = \psi_i \; \forall i \in I$; moreover, this property determines $\sum_{i \in I} A_i$ uniquely up to isomorphism. In particular, $\sum_{i \in I} A_i$ is a coproduct in the category $\mathcal{A}$ of abelian groups.

Proof. First, note that once we show that $\sum_{i \in I} A_i$ satisfies the property:

Theorem (8.5)

Let $\{A_i \mid i \in I\}$ be a family of abelian groups, written additively. If B is an abelian group and if $\{\psi_i : A_i \to B \mid i \in I\}$ is a family of homomorphisms, then there is a unique homomorphism $\psi : \sum_{i \in I} A_i \to B$ such that $\psi \iota_i = \psi_i \; \forall i \in I$; moreover, this property determines $\sum_{i \in I} A_i$ uniquely up to isomorphism. In particular, $\sum_{i \in I} A_i$ is a coproduct in the category $\mathcal{A}$ of abelian groups.

Proof. First, note that once we show that $\sum_{i \in I} A_i$ satisfies the property: for any abelian group B and family of homomorphisms $\{\psi_i : A_i \to B \mid i \in I\}$, there exists a unique homomorphism $\psi : \sum_{i \in I} A_i \to B$ such that $\psi \iota_i = \psi_i \; \forall i \in I$,

Theorem (8.5)

Let $\{A_i \mid i \in I\}$ be a family of abelian groups, written additively. If B is an abelian group and if $\{\psi_i : A_i \to B \mid i \in I\}$ is a family of homomorphisms, then there is a unique homomorphism $\psi : \sum_{i \in I} A_i \to B$ such that $\psi \iota_i = \psi_i \; \forall i \in I$; moreover, this property determines $\sum_{i \in I} A_i$ uniquely up to isomorphism. In particular, $\sum_{i \in I} A_i$ is a coproduct in the category $\mathcal{A}$ of abelian groups.

Proof. First, note that once we show that $\sum_{i \in I} A_i$ satisfies the property: for any abelian group B and family of homomorphisms $\{\psi_i : A_i \to B \mid i \in I\}$, there exists a unique homomorphism $\psi : \sum_{i \in I} A_i \to B$ such that $\psi \iota_i = \psi_i \; \forall i \in I$, we see that $\sum_{i \in I} A_i$ is a coproduct in the category $\mathcal{A}$ of abelian groups,

Theorem (8.5)

Let $\{A_i \mid i \in I\}$ be a family of abelian groups, written additively. If B is an abelian group and if $\{\psi_i : A_i \to B \mid i \in I\}$ is a family of homomorphisms, then there is a unique homomorphism $\psi : \sum_{i \in I} A_i \to B$ such that $\psi \iota_i = \psi_i \; \forall i \in I$; moreover, this property determines $\sum_{i \in I} A_i$ uniquely up to isomorphism. In particular, $\sum_{i \in I} A_i$ is a coproduct in the category $\mathcal{A}$ of abelian groups.

Proof. First, note that once we show that $\sum_{i \in I} A_i$ satisfies the property: for any abelian group B and family of homomorphisms $\{\psi_i : A_i \to B \mid i \in I\}$, there exists a unique homomorphism $\psi : \sum_{i \in I} A_i \to B$ such that $\psi \iota_i = \psi_i \; \forall i \in I$, we see that $\sum_{i \in I} A_i$ is a coproduct in the category $\mathcal{A}$ of abelian groups, and thus $\sum_{i \in I} A_i$ is determined uniquely up to isomorphism,

Theorem (8.5)

Let $\{A_i \mid i \in I\}$ be a family of abelian groups, written additively. If B is an abelian group and if $\{\psi_i : A_i \to B \mid i \in I\}$ is a family of homomorphisms, then there is a unique homomorphism $\psi : \sum_{i\in I} A_i \to B$ such that $\psi\iota_i = \psi_i\ \forall i \in I$; moreover, this property determines $\sum_{i\in I} A_i$ uniquely up to isomorphism. In particular, $\sum_{i\in I} A_i$ is a coproduct in the category $\mathcal{A}$ of abelian groups.

Proof. First, note that once we show that $\sum_{i\in I} A_i$ satisfies the property: for any abelian group B and family of homomorphisms $\{\psi_i : A_i \to B \mid i \in I\}$, there exists a unique homomorphism $\psi : \sum_{i\in I} A_i \to B$ such that $\psi\iota_i = \psi_i\ \forall i \in I$, we see that $\sum_{i\in I} A_i$ is a coproduct in the category $\mathcal{A}$ of abelian groups, and thus $\sum_{i\in I} A_i$ is determined uniquely up to isomorphism, by Theorem 7.5.

Theorem (8.5)

Let $\{A_i \mid i \in I\}$ be a family of abelian groups, written additively. If B is an abelian group and if $\{\psi_i : A_i \to B \mid i \in I\}$ is a family of homomorphisms, then there is a unique homomorphism $\psi : \sum_{i\in I} A_i \to B$ such that $\psi\iota_i = \psi_i \ \forall i \in I$; moreover, this property determines $\sum_{i\in I} A_i$ uniquely up to isomorphism. In particular, $\sum_{i\in I} A_i$ is a coproduct in the category $\mathcal{A}$ of abelian groups.

Proof. Therefore,

Theorem (8.5)

Let $\{A_i \mid i \in I\}$ be a family of abelian groups, written additively. If B is an abelian group and if $\{\psi_i : A_i \to B \mid i \in I\}$ is a family of homomorphisms, then there is a unique homomorphism $\psi : \sum_{i \in I} A_i \to B$ such that $\psi \iota_i = \psi_i \; \forall i \in I$; moreover, this property determines $\sum_{i \in I} A_i$ uniquely up to isomorphism. In particular, $\sum_{i \in I} A_i$ is a coproduct in the category $\mathcal{A}$ of abelian groups.

Proof. Therefore, it remains to show that $\sum_{i \in I} A_i$ satisfies the property: for any abelian group B and family of homomorphisms $\{\psi_i : A_i \to B \mid i \in I\}$, there exists a unique homomorphism $\psi : \sum_{i \in I} A_i \to B$ such that $\psi \iota_i = \psi_i \; \forall i \in I$.

Theorem (8.5)

Let $\{A_i \mid i \in I\}$ be a family of abelian groups, written additively. If B is an abelian group and if $\{\psi_i : A_i \to B \mid i \in I\}$ is a family of homomorphisms, then there is a unique homomorphism $\psi : \sum_{i \in I} A_i \to B$ such that $\psi \iota_i = \psi_i \ \forall i \in I$; moreover, this property determines $\sum_{i \in I} A_i$ uniquely up to isomorphism. In particular, $\sum_{i \in I} A_i$ is a coproduct in the category $\mathcal{A}$ of abelian groups.

Proof. Therefore, it remains to show that $\sum_{i \in I} A_i$ satisfies the property: for any abelian group B and family of homomorphisms $\{\psi_i : A_i \to B \mid i \in I\}$, there exists a unique homomorphism $\psi : \sum_{i \in I} A_i \to B$ such that $\psi \iota_i = \psi_i \ \forall i \in I$. Through out this proof,

Theorem (8.5)

Let $\{A_i \mid i \in I\}$ be a family of abelian groups, written additively. If B is an abelian group and if $\{\psi_i : A_i \to B \mid i \in I\}$ is a family of homomorphisms, then there is a unique homomorphism $\psi : \sum_{i \in I} A_i \to B$ such that $\psi \iota_i = \psi_i \ \forall i \in I$; moreover, this property determines $\sum_{i \in I} A_i$ uniquely up to isomorphism. In particular, $\sum_{i \in I} A_i$ is a coproduct in the category $\mathcal{A}$ of abelian groups.

Proof. Therefore, it remains to show that $\sum_{i \in I} A_i$ satisfies the property: for any abelian group B and family of homomorphisms $\{\psi_i : A_i \to B \mid i \in I\}$, there exists a unique homomorphism $\psi : \sum_{i \in I} A_i \to B$ such that $\psi \iota_i = \psi_i \ \forall i \in I$. Through out this proof, all groups will be written additively,

Theorem (8.5)

Let $\{A_i \mid i \in I\}$ be a family of abelian groups, written additively. If B is an abelian group and if $\{\psi_i : A_i \to B \mid i \in I\}$ is a family of homomorphisms, then there is a unique homomorphism $\psi : \sum_{i \in I} A_i \to B$ such that $\psi \iota_i = \psi_i \; \forall i \in I$; moreover, this property determines $\sum_{i \in I} A_i$ uniquely up to isomorphism. In particular, $\sum_{i \in I} A_i$ is a coproduct in the category $\mathcal{A}$ of abelian groups.

Proof. Therefore, it remains to show that $\sum_{i \in I} A_i$ satisfies the property: for any abelian group B and family of homomorphisms $\{\psi_i : A_i \to B \mid i \in I\}$, there exists a unique homomorphism $\psi : \sum_{i \in I} A_i \to B$ such that $\psi \iota_i = \psi_i \; \forall i \in I$.
Through out this proof, all groups will be written additively, i.e., we write "$+$" for the binary operation,

Theorem (8.5)

Let $\{A_i \mid i \in I\}$ be a family of abelian groups, written additively. If B is an abelian group and if $\{\psi_i : A_i \to B \mid i \in I\}$ is a family of homomorphisms, then there is a unique homomorphism $\psi : \sum_{i \in I} A_i \to B$ such that $\psi \iota_i = \psi_i \ \forall i \in I$; moreover, this property determines $\sum_{i \in I} A_i$ uniquely up to isomorphism. In particular, $\sum_{i \in I} A_i$ is a coproduct in the category $\mathcal{A}$ of abelian groups.

Proof. Therefore, it remains to show that $\sum_{i \in I} A_i$ satisfies the property: for any abelian group B and family of homomorphisms $\{\psi_i : A_i \to B \mid i \in I\}$, there exists a unique homomorphism $\psi : \sum_{i \in I} A_i \to B$ such that $\psi \iota_i = \psi_i \ \forall i \in I$. Through out this proof, all groups will be written additively, i.e., we write "$+$" for the binary operation, "0" for the identity element,

Theorem (8.5)

Let $\{A_i \mid i \in I\}$ be a family of abelian groups, written additively. If B is an abelian group and if $\{\psi_i : A_i \to B \mid i \in I\}$ is a family of homomorphisms, then there is a unique homomorphism $\psi : \sum_{i \in I} A_i \to B$ such that $\psi \iota_i = \psi_i \; \forall i \in I$; moreover, this property determines $\sum_{i \in I} A_i$ uniquely up to isomorphism. In particular, $\sum_{i \in I} A_i$ is a coproduct in the category $\mathcal{A}$ of abelian groups.

Proof. Therefore, it remains to show that $\sum_{i \in I} A_i$ satisfies the property: for any abelian group B and family of homomorphisms $\{\psi_i : A_i \to B \mid i \in I\}$, there exists a unique homomorphism $\psi : \sum_{i \in I} A_i \to B$ such that $\psi \iota_i = \psi_i \; \forall i \in I$. Through out this proof, all groups will be written additively, i.e., we write "$+$" for the binary operation, "0" for the identity element, and "$-a$" for the inverse of a.

Proof of Theorem (8.5)

Let B be an abelian group

Proof of Theorem (8.5)

Let B be an abelian group and let $\{\psi_i : A_i \to B \mid i \in I\}$ be a family of homomorphisms.

Proof of Theorem (8.5)

Let B be an abelian group and let $\{\psi_i : A_i \to B \mid i \in I\}$ be a family of homomorphisms. We claim that there exists a unique homomorphism $\psi : \sum_{i \in I} A_i \to B$

Proof of Theorem (8.5)

Let B be an abelian group and let $\{\psi_i : A_i \to B \mid i \in I\}$ be a family of homomorphisms. We claim that there exists a unique homomorphism $\psi : \sum_{i\in I} A_i \to B$ such that $\psi\iota_i = \psi_i \; \forall i \in I$.

Proof of Theorem (8.5)

Let B be an abelian group and let $\{\psi_i : A_i \to B \mid i \in I\}$ be a family of homomorphisms. We claim that there exists a unique homomorphism $\psi : \sum_{i \in I} A_i \to B$ such that $\psi \iota_i = \psi_i \; \forall i \in I$.

Note that if $(a_i)_{i \in I} \in \sum_{i \in I} A_i$ is nonzero,

Proof of Theorem (8.5)

Let B be an abelian group and let $\{\psi_i : A_i \to B \mid i \in I\}$ be a family of homomorphisms. We claim that there exists a unique homomorphism $\psi : \sum_{i\in I} A_i \to B$ such that $\psi \iota_i = \psi_i \; \forall i \in I$.

Note that if $(a_i)_{i\in I} \in \sum_{i\in I} A_i$ is nonzero, then there are only finitely many nonzero a_i,

Proof of Theorem (8.5)

Let B be an abelian group and let $\{\psi_i : A_i \to B \mid i \in I\}$ be a family of homomorphisms. We claim that there exists a unique homomorphism $\psi : \sum_{i\in I} A_i \to B$ such that $\psi\iota_i = \psi_i \; \forall i \in I$.

Note that if $(a_i)_{i\in I} \in \sum_{i\in I} A_i$ is nonzero, then there are only finitely many nonzero a_i, so $I_{\neq 0} := \{i \in I \mid a_i \neq 0\}$ is finite.

Proof of Theorem (8.5)

Let B be an abelian group and let $\{\psi_i : A_i \to B \mid i \in I\}$ be a family of homomorphisms. We claim that there exists a unique homomorphism $\psi : \sum_{i \in I} A_i \to B$ such that $\psi \iota_i = \psi_i \ \forall i \in I$.

Note that if $(a_i)_{i \in I} \in \sum_{i \in I} A_i$ is nonzero, then there are only finitely many nonzero a_i, so $I_{\neq 0} := \{i \in I \mid a_i \neq 0\}$ is finite.

Existence:

Proof of Theorem (8.5)

Let B be an abelian group and let $\{\psi_i : A_i \to B \mid i \in I\}$ be a family of homomorphisms. We claim that there exists a unique homomorphism $\psi : \sum_{i\in I} A_i \to B$ such that $\psi\iota_i = \psi_i \; \forall i \in I$.

Note that if $(a_i)_{i\in I} \in \sum_{i\in I} A_i$ is nonzero, then there are only finitely many nonzero a_i, so $I_{\neq 0} := \{i \in I \mid a_i \neq 0\}$ is finite.

Existence: Define $\psi : \sum_{i\in I} A_i \to B$ as follows:

Proof of Theorem (8.5)

Let B be an abelian group and let $\{\psi_i : A_i \to B \mid i \in I\}$ be a family of homomorphisms. We claim that there exists a unique homomorphism $\psi : \sum_{i\in I} A_i \to B$ such that $\psi\iota_i = \psi_i \; \forall i \in I$.

Note that if $(a_i)_{i\in I} \in \sum_{i\in I} A_i$ is nonzero, then there are only finitely many nonzero a_i, so $I_{\neq 0} := \{i \in I \mid a_i \neq 0\}$ is finite.

Existence: Define $\psi : \sum_{i\in I} A_i \to B$ as follows: $\psi(0) = 0$

Proof of Theorem (8.5)

Let B be an abelian group and let $\{\psi_i : A_i \to B \mid i \in I\}$ be a family of homomorphisms. We claim that there exists a unique homomorphism $\psi : \sum_{i\in I} A_i \to B$ such that $\psi\iota_i = \psi_i \; \forall i \in I$.

Note that if $(a_i)_{i\in I} \in \sum_{i\in I} A_i$ is nonzero, then there are only finitely many nonzero a_i, so $I_{\neq 0} := \{i \in I \mid a_i \neq 0\}$ is finite.

Existence: Define $\psi : \sum_{i\in I} A_i \to B$ as follows: $\psi(0) = 0$ and for nonzero $(a_i)_{i\in I} \in \sum_{i\in I} A_i$,

Proof of Theorem (8.5)

Let B be an abelian group and let $\{\psi_i : A_i \to B \mid i \in I\}$ be a family of homomorphisms. We claim that there exists a unique homomorphism $\psi : \sum_{i \in I} A_i \to B$ such that $\psi \iota_i = \psi_i \; \forall i \in I$.

Note that if $(a_i)_{i \in I} \in \sum_{i \in I} A_i$ is nonzero, then there are only finitely many nonzero a_i, so $I_{\neq 0} := \{i \in I \mid a_i \neq 0\}$ is finite.

Existence: Define $\psi : \sum_{i \in I} A_i \to B$ as follows: $\psi(0) = 0$ and for nonzero $(a_i)_{i \in I} \in \sum_{i \in I} A_i$, if $I_{\neq 0} = \{i_1, i_2, \ldots, i_r\}$,

Proof of Theorem (8.5)

Let B be an abelian group and let $\{\psi_i : A_i \to B \mid i \in I\}$ be a family of homomorphisms. We claim that there exists a unique homomorphism $\psi : \sum_{i\in I} A_i \to B$ such that $\psi\iota_i = \psi_i \ \forall i \in I$.

Note that if $(a_i)_{i\in I} \in \sum_{i\in I} A_i$ is nonzero, then there are only finitely many nonzero a_i, so $I_{\neq 0} := \{i \in I \mid a_i \neq 0\}$ is finite.

Existence: Define $\psi : \sum_{i\in I} A_i \to B$ as follows: $\psi(0) = 0$ and for nonzero $(a_i)_{i\in I} \in \sum_{i\in I} A_i$, if $I_{\neq 0} = \{i_1, i_2, \ldots, i_r\}$,

$$\psi\big((a_i)_{i\in I}\big)$$

Proof of Theorem (8.5)

Let B be an abelian group and let $\{\psi_i : A_i \to B \mid i \in I\}$ be a family of homomorphisms. We claim that there exists a unique homomorphism $\psi : \sum_{i \in I} A_i \to B$ such that $\psi \iota_i = \psi_i \; \forall i \in I$.

Note that if $(a_i)_{i \in I} \in \sum_{i \in I} A_i$ is nonzero, then there are only finitely many nonzero a_i, so $I_{\neq 0} := \{i \in I \mid a_i \neq 0\}$ is finite.

Existence: Define $\psi : \sum_{i \in I} A_i \to B$ as follows: $\psi(0) = 0$ and for nonzero $(a_i)_{i \in I} \in \sum_{i \in I} A_i$, if $I_{\neq 0} = \{i_1, i_2, \ldots, i_r\}$,

$$\psi\big((a_i)_{i \in I}\big) = \psi_{i_1}(a_{i_1}) + \psi_{i_2}(a_{i_2}) + \cdots + \psi_{i_r}(a_{i_r})$$

Proof of Theorem (8.5)

Let B be an abelian group and let $\{\psi_i : A_i \to B \mid i \in I\}$ be a family of homomorphisms. We claim that there exists a unique homomorphism $\psi : \sum_{i \in I} A_i \to B$ such that $\psi \iota_i = \psi_i \; \forall i \in I$.

Note that if $(a_i)_{i \in I} \in \sum_{i \in I} A_i$ is nonzero, then there are only finitely many nonzero a_i, so $I_{\neq 0} := \{i \in I \mid a_i \neq 0\}$ is finite.

Existence: Define $\psi : \sum_{i \in I} A_i \to B$ as follows: $\psi(0) = 0$ and for nonzero $(a_i)_{i \in I} \in \sum_{i \in I} A_i$, if $I_{\neq 0} = \{i_1, i_2, \ldots, i_r\}$,

$$\begin{aligned} \psi\big((a_i)_{i \in I}\big) &= \psi_{i_1}(a_{i_1}) + \psi_{i_2}(a_{i_2}) + \cdots + \psi_{i_r}(a_{i_r}) \\ &= \textstyle\sum_{i \in I_{\neq 0}} \psi_i(a_i). \end{aligned}$$

Proof of Theorem (8.5)

Let B be an abelian group and let $\{\psi_i : A_i \to B \mid i \in I\}$ be a family of homomorphisms. We claim that there exists a unique homomorphism $\psi : \sum_{i \in I} A_i \to B$ such that $\psi\iota_i = \psi_i \; \forall i \in I$.

Note that if $(a_i)_{i \in I} \in \sum_{i \in I} A_i$ is nonzero, then there are only finitely many nonzero a_i, so $I_{\neq 0} := \{i \in I \mid a_i \neq 0\}$ is finite.

Existence: Define $\psi : \sum_{i \in I} A_i \to B$ as follows: $\psi(0) = 0$ and for nonzero $(a_i)_{i \in I} \in \sum_{i \in I} A_i$, if $I_{\neq 0} = \{i_1, i_2, \ldots, i_r\}$,

$$\begin{aligned} \psi\big((a_i)_{i \in I}\big) &= \psi_{i_1}(a_{i_1}) + \psi_{i_2}(a_{i_2}) + \cdots + \psi_{i_r}(a_{i_r}) \\ &= \textstyle\sum_{i \in I_{\neq 0}} \psi_i(a_i). \end{aligned}$$

Note that because B is abelian,

Proof of Theorem (8.5)

Let B be an abelian group and let $\{\psi_i : A_i \to B \mid i \in I\}$ be a family of homomorphisms. We claim that there exists a unique homomorphism $\psi : \sum_{i\in I} A_i \to B$ such that $\psi\iota_i = \psi_i \; \forall i \in I$.

Note that if $(a_i)_{i\in I} \in \sum_{i\in I} A_i$ is nonzero, then there are only finitely many nonzero a_i, so $I_{\neq 0} := \{i \in I \mid a_i \neq 0\}$ is finite.

Existence: Define $\psi : \sum_{i\in I} A_i \to B$ as follows: $\psi(0) = 0$ and for nonzero $(a_i)_{i\in I} \in \sum_{i\in I} A_i$, if $I_{\neq 0} = \{i_1, i_2, \ldots, i_r\}$,

$$\begin{aligned} \psi\big((a_i)_{i\in I}\big) &= \psi_{i_1}(a_{i_1}) + \psi_{i_2}(a_{i_2}) + \cdots + \psi_{i_r}(a_{i_r}) \\ &= \textstyle\sum_{i\in I_{\neq 0}} \psi_i(a_i). \end{aligned}$$

Note that because B is abelian, the notation $\sum_{i\in I_{\neq 0}} \psi_i(a_i)$ makes sense.

Proof of Theorem (8.5)

Let B be an abelian group and let $\{\psi_i : A_i \to B \mid i \in I\}$ be a family of homomorphisms. We claim that there exists a unique homomorphism $\psi : \sum_{i \in I} A_i \to B$ such that $\psi\iota_i = \psi_i \; \forall i \in I$.

Note that if $(a_i)_{i \in I} \in \sum_{i \in I} A_i$ is nonzero, then there are only finitely many nonzero a_i, so $I_{\neq 0} := \{i \in I \mid a_i \neq 0\}$ is finite.

Existence: Define $\psi : \sum_{i \in I} A_i \to B$ as follows: $\psi(0) = 0$ and for nonzero $(a_i)_{i \in I} \in \sum_{i \in I} A_i$, if $I_{\neq 0} = \{i_1, i_2, \ldots, i_r\}$,

$$\begin{aligned} \psi\big((a_i)_{i \in I}\big) &= \psi_{i_1}(a_{i_1}) + \psi_{i_2}(a_{i_2}) + \cdots + \psi_{i_r}(a_{i_r}) \\ &= \textstyle\sum_{i \in I_{\neq 0}} \psi_i(a_i). \end{aligned}$$

Note that because B is abelian, the notation $\sum_{i \in I_{\neq 0}} \psi_i(a_i)$ makes sense. Also, since B is abelian,

Proof of Theorem (8.5)

Let B be an abelian group and let $\{\psi_i : A_i \to B \mid i \in I\}$ be a family of homomorphisms. We claim that there exists a unique homomorphism $\psi : \sum_{i\in I} A_i \to B$ such that $\psi\iota_i = \psi_i \ \forall i \in I$.

Note that if $(a_i)_{i\in I} \in \sum_{i\in I} A_i$ is nonzero, then there are only finitely many nonzero a_i, so $I_{\neq 0} := \{i \in I \mid a_i \neq 0\}$ is finite.

Existence: Define $\psi : \sum_{i\in I} A_i \to B$ as follows: $\psi(0) = 0$ and for nonzero $(a_i)_{i\in I} \in \sum_{i\in I} A_i$, if $I_{\neq 0} = \{i_1, i_2, \ldots, i_r\}$,

$$\begin{aligned}\psi\big((a_i)_{i\in I}\big) &= \psi_{i_1}(a_{i_1}) + \psi_{i_2}(a_{i_2}) + \cdots + \psi_{i_r}(a_{i_r}) \\ &= \textstyle\sum_{i\in I_{\neq 0}} \psi_i(a_i).\end{aligned}$$

Note that because B is abelian, the notation $\sum_{i\in I_{\neq 0}} \psi_i(a_i)$ makes sense. Also, since B is abelian, it is easy to see that ψ is a homomorphism.

Proof of Theorem (8.5)

Let B be an abelian group and let $\{\psi_i : A_i \to B \mid i \in I\}$ be a family of homomorphisms. We claim that there exists a unique homomorphism $\psi : \sum_{i\in I} A_i \to B$ such that $\psi\iota_i = \psi_i \; \forall i \in I$.

Note that if $(a_i)_{i\in I} \in \sum_{i\in I} A_i$ is nonzero, then there are only finitely many nonzero a_i, so $I_{\neq 0} := \{i \in I \mid a_i \neq 0\}$ is finite.

Existence: Define $\psi : \sum_{i\in I} A_i \to B$ as follows: $\psi(0) = 0$ and for nonzero $(a_i)_{i\in I} \in \sum_{i\in I} A_i$, if $I_{\neq 0} = \{i_1, i_2, \ldots, i_r\}$,

$$\begin{aligned} \psi\big((a_i)_{i\in I}\big) &= \psi_{i_1}(a_{i_1}) + \psi_{i_2}(a_{i_2}) + \cdots + \psi_{i_r}(a_{i_r}) \\ &= \textstyle\sum_{i\in I_{\neq 0}} \psi_i(a_i). \end{aligned}$$

Note that because B is abelian, the notation $\sum_{i\in I_{\neq 0}} \psi_i(a_i)$ makes sense. Also, since B is abelian, it is easy to see that ψ is a homomorphism. It is also easy to verify that $\psi\iota_i = \psi_i \; \forall i \in I$.

Proof of Theorem (8.5)

Let B be an abelian group and let $\{\psi_i : A_i \to B \mid i \in I\}$ be a family of homomorphisms. We claim that there exists a unique homomorphism $\psi : \sum_{i \in I} A_i \to B$ such that $\psi\iota_i = \psi_i \; \forall i \in I$.

Note that if $(a_i)_{i \in I} \in \sum_{i \in I} A_i$ is nonzero, then there are only finitely many nonzero a_i, so $I_{\neq 0} := \{i \in I \mid a_i \neq 0\}$ is finite.

Uniqueness:

Proof of Theorem (8.5)

Let B be an abelian group and let $\{\psi_i : A_i \to B \mid i \in I\}$ be a family of homomorphisms. We claim that there exists a unique homomorphism $\psi : \sum_{i \in I} A_i \to B$ such that $\psi \iota_i = \psi_i \ \forall i \in I$.

Note that if $(a_i)_{i \in I} \in \sum_{i \in I} A_i$ is nonzero, then there are only finitely many nonzero a_i, so $I_{\neq 0} := \{i \in I \mid a_i \neq 0\}$ is finite.

Uniqueness: Assume $\varphi : \sum_{i \in I} A_i \to B$ is a homomorphism

Proof of Theorem (8.5)

Let B be an abelian group and let $\{\psi_i : A_i \to B \mid i \in I\}$ be a family of homomorphisms. We claim that there exists a unique homomorphism $\psi : \sum_{i \in I} A_i \to B$ such that $\psi \iota_i = \psi_i \; \forall i \in I$.

Note that if $(a_i)_{i \in I} \in \sum_{i \in I} A_i$ is nonzero, then there are only finitely many nonzero a_i, so $I_{\neq 0} := \{i \in I \mid a_i \neq 0\}$ is finite.

Uniqueness: Assume $\varphi : \sum_{i \in I} A_i \to B$ is a homomorphism such that $\varphi \iota_i = \psi_i \; \forall i \in I$.

Proof of Theorem (8.5)

Let B be an abelian group and let $\{\psi_i : A_i \to B \mid i \in I\}$ be a family of homomorphisms. We claim that there exists a unique homomorphism $\psi : \sum_{i\in I} A_i \to B$ such that $\psi\iota_i = \psi_i \; \forall i \in I$.

Note that if $(a_i)_{i\in I} \in \sum_{i\in I} A_i$ is nonzero, then there are only finitely many nonzero a_i, so $I_{\neq 0} := \{i \in I \mid a_i \neq 0\}$ is finite.

Uniqueness: Assume $\varphi : \sum_{i\in I} A_i \to B$ is a homomorphism such that $\varphi\iota_i = \psi_i \; \forall i \in I$. First, note that if $(a_i)_{i\in I} \in \sum_{i\in I} A_i$ is nonzero,

Proof of Theorem (8.5)

Let B be an abelian group and let $\{\psi_i : A_i \to B \mid i \in I\}$ be a family of homomorphisms. We claim that there exists a unique homomorphism $\psi : \sum_{i \in I} A_i \to B$ such that $\psi \iota_i = \psi_i \; \forall i \in I$.

Note that if $(a_i)_{i \in I} \in \sum_{i \in I} A_i$ is nonzero, then there are only finitely many nonzero a_i, so $I_{\neq 0} := \{i \in I \mid a_i \neq 0\}$ is finite.

Uniqueness: Assume $\varphi : \sum_{i \in I} A_i \to B$ is a homomorphism such that $\varphi \iota_i = \psi_i \; \forall i \in I$. First, note that if $(a_i)_{i \in I} \in \sum_{i \in I} A_i$ is nonzero, $(a_i)_{i \in I} = \sum_{i \in I_{\neq 0}} \iota_i(a_i)$.

Proof of Theorem (8.5)

Let B be an abelian group and let $\{\psi_i : A_i \to B \mid i \in I\}$ be a family of homomorphisms. We claim that there exists a unique homomorphism $\psi : \sum_{i\in I} A_i \to B$ such that $\psi\iota_i = \psi_i \; \forall i \in I$.

Note that if $(a_i)_{i\in I} \in \sum_{i\in I} A_i$ is nonzero, then there are only finitely many nonzero a_i, so $I_{\neq 0} := \{i \in I \mid a_i \neq 0\}$ is finite.

Uniqueness: Assume $\varphi : \sum_{i\in I} A_i \to B$ is a homomorphism such that $\varphi\iota_i = \psi_i \; \forall i \in I$. First, note that if $(a_i)_{i\in I} \in \sum_{i\in I} A_i$ is nonzero, $(a_i)_{i\in I} = \sum_{i\in I_{\neq 0}} \iota_i(a_i)$. Therefore,

Proof of Theorem (8.5)

Let B be an abelian group and let $\{\psi_i : A_i \to B \mid i \in I\}$ be a family of homomorphisms. We claim that there exists a unique homomorphism $\psi : \sum_{i\in I} A_i \to B$ such that $\psi\iota_i = \psi_i \; \forall i \in I$.

Note that if $(a_i)_{i\in I} \in \sum_{i\in I} A_i$ is nonzero, then there are only finitely many nonzero a_i, so $I_{\neq 0} := \{i \in I \mid a_i \neq 0\}$ is finite.

Uniqueness: Assume $\varphi : \sum_{i\in I} A_i \to B$ is a homomorphism such that $\varphi\iota_i = \psi_i \; \forall i \in I$. First, note that if $(a_i)_{i\in I} \in \sum_{i\in I} A_i$ is nonzero, $(a_i)_{i\in I} = \sum_{i\in I_{\neq 0}} \iota_i(a_i)$. Therefore,

$$\varphi\big((a_i)_{i\in I}\big)$$

Proof of Theorem (8.5)

Let B be an abelian group and let $\{\psi_i : A_i \to B \mid i \in I\}$ be a family of homomorphisms. We claim that there exists a unique homomorphism $\psi : \sum_{i \in I} A_i \to B$ such that $\psi \iota_i = \psi_i \; \forall i \in I$.

Note that if $(a_i)_{i \in I} \in \sum_{i \in I} A_i$ is nonzero, then there are only finitely many nonzero a_i, so $I_{\neq 0} := \{i \in I \mid a_i \neq 0\}$ is finite.

Uniqueness: Assume $\varphi : \sum_{i \in I} A_i \to B$ is a homomorphism such that $\varphi \iota_i = \psi_i \; \forall i \in I$. First, note that if $(a_i)_{i \in I} \in \sum_{i \in I} A_i$ is nonzero, $(a_i)_{i \in I} = \sum_{i \in I_{\neq 0}} \iota_i(a_i)$. Therefore,

$$\varphi\big((a_i)_{i \in I}\big) \;=\; \varphi\Big(\sum_{i \in I_{\neq 0}} \iota_i(a_i)\Big)$$

Proof of Theorem (8.5)

Let B be an abelian group and let $\{\psi_i : A_i \to B \mid i \in I\}$ be a family of homomorphisms. We claim that there exists a unique homomorphism $\psi : \sum_{i \in I} A_i \to B$ such that $\psi\iota_i = \psi_i \; \forall i \in I$.

Note that if $(a_i)_{i \in I} \in \sum_{i \in I} A_i$ is nonzero, then there are only finitely many nonzero a_i, so $I_{\neq 0} := \{i \in I \mid a_i \neq 0\}$ is finite.

Uniqueness: Assume $\varphi : \sum_{i \in I} A_i \to B$ is a homomorphism such that $\varphi\iota_i = \psi_i \; \forall i \in I$. First, note that if $(a_i)_{i \in I} \in \sum_{i \in I} A_i$ is nonzero, $(a_i)_{i \in I} = \sum_{i \in I_{\neq 0}} \iota_i(a_i)$. Therefore,

$$\varphi\big((a_i)_{i \in I}\big) = \varphi\Big(\sum_{i \in I_{\neq 0}} \iota_i(a_i)\Big) = \sum_{i \in I_{\neq 0}} \varphi\big(\iota_i(a_i)\big)$$

Proof of Theorem (8.5)

Let B be an abelian group and let $\{\psi_i : A_i \to B \mid i \in I\}$ be a family of homomorphisms. We claim that there exists a unique homomorphism $\psi : \sum_{i\in I} A_i \to B$ such that $\psi\iota_i = \psi_i \; \forall i \in I$.

Note that if $(a_i)_{i\in I} \in \sum_{i\in I} A_i$ is nonzero, then there are only finitely many nonzero a_i, so $I_{\neq 0} := \{i \in I \mid a_i \neq 0\}$ is finite.

Uniqueness: Assume $\varphi : \sum_{i\in I} A_i \to B$ is a homomorphism such that $\varphi\iota_i = \psi_i \; \forall i \in I$. First, note that if $(a_i)_{i\in I} \in \sum_{i\in I} A_i$ is nonzero, $(a_i)_{i\in I} = \sum_{i\in I_{\neq 0}} \iota_i(a_i)$. Therefore,

$$\begin{aligned} \varphi\big((a_i)_{i\in I}\big) &= \varphi\Big(\sum_{i\in I_{\neq 0}} \iota_i(a_i)\Big) = \sum_{i\in I_{\neq 0}} \varphi\big(\iota_i(a_i)\big) \\ &= \sum_{i\in I_{\neq 0}} \psi_i(a_i) \end{aligned}$$

Proof of Theorem (8.5)

Let B be an abelian group and let $\{\psi_i : A_i \to B \mid i \in I\}$ be a family of homomorphisms. We claim that there exists a unique homomorphism $\psi : \sum_{i\in I} A_i \to B$ such that $\psi\iota_i = \psi_i \; \forall i \in I$.

Note that if $(a_i)_{i\in I} \in \sum_{i\in I} A_i$ is nonzero, then there are only finitely many nonzero a_i, so $I_{\neq 0} := \{i \in I \mid a_i \neq 0\}$ is finite.

Uniqueness: Assume $\varphi : \sum_{i\in I} A_i \to B$ is a homomorphism such that $\varphi\iota_i = \psi_i \; \forall i \in I$. First, note that if $(a_i)_{i\in I} \in \sum_{i\in I} A_i$ is nonzero, $(a_i)_{i\in I} = \sum_{i\in I_{\neq 0}} \iota_i(a_i)$. Therefore,

$$\begin{aligned}
\varphi\big((a_i)_{i\in I}\big) &= \varphi\Big(\sum_{i\in I_{\neq 0}} \iota_i(a_i)\Big) &&= \sum_{i\in I_{\neq 0}} \varphi\big(\iota_i(a_i)\big) \\
&= \sum_{i\in I_{\neq 0}} \psi_i(a_i) &&= \sum_{i\in I_{\neq 0}} \psi\big(\iota_i(a_i)\big)
\end{aligned}$$

Proof of Theorem (8.5)

Let B be an abelian group and let $\{\psi_i : A_i \to B \mid i \in I\}$ be a family of homomorphisms. We claim that there exists a unique homomorphism $\psi : \sum_{i\in I} A_i \to B$ such that $\psi\iota_i = \psi_i \; \forall i \in I$.

Note that if $(a_i)_{i\in I} \in \sum_{i\in I} A_i$ is nonzero, then there are only finitely many nonzero a_i, so $I_{\neq 0} := \{i \in I \mid a_i \neq 0\}$ is finite.

Uniqueness: Assume $\varphi : \sum_{i\in I} A_i \to B$ is a homomorphism such that $\varphi\iota_i = \psi_i \; \forall i \in I$. First, note that if $(a_i)_{i\in I} \in \sum_{i\in I} A_i$ is nonzero, $(a_i)_{i\in I} = \sum_{i\in I_{\neq 0}} \iota_i(a_i)$. Therefore,

$$\begin{aligned}
\varphi\big((a_i)_{i\in I}\big) &= \varphi\Big(\sum_{i\in I_{\neq 0}} \iota_i(a_i)\Big) &&= \sum_{i\in I_{\neq 0}} \varphi\big(\iota_i(a_i)\big) \\
&= \sum_{i\in I_{\neq 0}} \psi_i(a_i) &&= \sum_{i\in I_{\neq 0}} \psi\big(\iota_i(a_i)\big) \\
&= \psi\Big(\sum_{i\in I_{\neq 0}} \iota_i(a_i)\Big)
\end{aligned}$$

Proof of Theorem (8.5)

Let B be an abelian group and let $\{\psi_i : A_i \to B \mid i \in I\}$ be a family of homomorphisms. We claim that there exists a unique homomorphism $\psi : \sum_{i \in I} A_i \to B$ such that $\psi \iota_i = \psi_i \ \forall i \in I$.

Note that if $(a_i)_{i \in I} \in \sum_{i \in I} A_i$ is nonzero, then there are only finitely many nonzero a_i, so $I_{\neq 0} := \{i \in I \mid a_i \neq 0\}$ is finite.

Uniqueness: Assume $\varphi : \sum_{i \in I} A_i \to B$ is a homomorphism such that $\varphi \iota_i = \psi_i \ \forall i \in I$. First, note that if $(a_i)_{i \in I} \in \sum_{i \in I} A_i$ is nonzero, $(a_i)_{i \in I} = \sum_{i \in I_{\neq 0}} \iota_i(a_i)$. Therefore,

$$\begin{aligned} \varphi\big((a_i)_{i \in I}\big) &= \varphi\Big(\sum_{i \in I_{\neq 0}} \iota_i(a_i)\Big) &&= \sum_{i \in I_{\neq 0}} \varphi\big(\iota_i(a_i)\big) \\ &= \sum_{i \in I_{\neq 0}} \psi_i(a_i) &&= \sum_{i \in I_{\neq 0}} \psi\big(\iota_i(a_i)\big) \\ &= \psi\Big(\sum_{i \in I_{\neq 0}} \iota_i(a_i)\Big) &&= \psi\big((a_i)_{i \in I}\big). \end{aligned}$$

Proof of Theorem (8.5)

Let B be an abelian group and let $\{\psi_i : A_i \to B \mid i \in I\}$ be a family of homomorphisms. We claim that there exists a unique homomorphism $\psi : \sum_{i\in I} A_i \to B$ such that $\psi\iota_i = \psi_i \; \forall i \in I$.

Note that if $(a_i)_{i\in I} \in \sum_{i\in I} A_i$ is nonzero, then there are only finitely many nonzero a_i, so $I_{\neq 0} := \{i \in I \mid a_i \neq 0\}$ is finite.

Uniqueness: Assume $\varphi : \sum_{i\in I} A_i \to B$ is a homomorphism such that $\varphi\iota_i = \psi_i \; \forall i \in I$. First, note that if $(a_i)_{i\in I} \in \sum_{i\in I} A_i$ is nonzero, $(a_i)_{i\in I} = \sum_{i\in I_{\neq 0}} \iota_i(a_i)$. Therefore,

$$\begin{aligned}
\varphi\big((a_i)_{i\in I}\big) &= \varphi\Big(\sum_{i\in I_{\neq 0}} \iota_i(a_i)\Big) &&= \sum_{i\in I_{\neq 0}} \varphi\big(\iota_i(a_i)\big) \\
&= \sum_{i\in I_{\neq 0}} \psi_i(a_i) &&= \sum_{i\in I_{\neq 0}} \psi\big(\iota_i(a_i)\big) \\
&= \psi\Big(\sum_{i\in I_{\neq 0}} \iota_i(a_i)\Big) &&= \psi\big((a_i)_{i\in I}\big).
\end{aligned}$$

Hence $\varphi = \psi$.