

Modern Algebra I

Lecture 4

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We will cover the following sections today:

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Section I.6: Symmetric, Alternating, and Dihedral Groups

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Section I.6: Symmetric, Alternating, and Dihedral Groups

Section I.7: Categories: Products, Coproducts, and
Free Objects

Chapter I: Groups

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Chapter I: Groups

Section 1.6: Symmetric, Alternating, and Dihedral Groups

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Chapter I: Groups

Section 1.6: Symmetric, Alternating, and Dihedral Groups

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Since we have shown these two lemmas last week, we are ready to prove Theorem (6.8) now.

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Proof. Note that in an abelian group, every subgroup is a normal subgroup. Hence, if G is a simple abelian group, then G has no proper subgroup. Thus, G must be cyclic, and so $G \simeq \mathbb{Z}$ or $G \simeq \mathbb{Z}_n$. It is clear that \mathbb{Z} has proper subgroup, so $G \simeq \mathbb{Z}$ is impossible.

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Theorem (6.13). For each $n \geq 3$, the dihedral group D_n is a group of order $2n$ whose generators a and b satisfy

(i) $a^n = e; b^2 = e; \text{ if } 0 < k < n, a^k \neq e.$

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Now, let's show that if $a^i b^j = a^s b^t$ with $0 \leq i, s \leq n - 1$ and $0 \leq j, t \leq 1$, then $i = s$ and $j = t$.

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Theorem (6.13). For each $n \geq 3$, the dihedral group D_n is a group of order $2n$ whose generators a and b satisfy

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We claim that $G \cong D_n$.

Similarly as we did for D_n , using (i) and (ii) for G , we have

$$G = \langle \alpha, \beta \rangle = \{ \alpha^i \beta^j \mid 0 \leq i \leq n-1, 0 \leq j \leq 1 \}.$$

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Proof of Theorem (6.13)

Suppose G is a group generated by $\alpha, \beta \in G$ satisfying that for some $n \geq 3$,

(i) $\alpha^n = e; \beta^2 = e; \text{ if } 0 < k < n, \alpha^k \neq e;$

(ii) $\beta\alpha = \alpha^{-1}\beta.$

We claim that $G \cong D_n$.

Similarly as we did for D_n , using (i) and (ii) for G , we have

$$G = \langle \alpha, \beta \rangle = \{ \alpha^i \beta^j \mid 0 \leq i \leq n-1, 0 \leq j \leq 1 \}.$$

Consider the map $f : D_n \rightarrow G$ with $a^i b^j \mapsto \alpha^i \beta^j$.

Thus, f is an epimorphism. Let $a^i b^j \in \text{Ker } f$ with $0 \leq i \leq n-1$ and $0 \leq j \leq 1$. Then $\alpha^i \beta^j = e$ in G .

Hence $j = 0$ and $\alpha^i = e \implies i = 0$. Therefore $\text{Ker } f = \{e\}$ and so f is an isomorphism.

Chapter I

Section I.7: Categories: Products, Coproducts, and Free Objects

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where $g \circ f$ is the composition of the maps f and g . Note that the composition of maps is associative and for every $B \in \mathcal{S}$, the identity map $1_B : B \rightarrow B$ satisfies the property that for any maps $f : A \rightarrow B$ and $g : B \rightarrow C$, $1_B \circ f = f$ and $g \circ 1_B = g$. Hence \mathcal{S} is a category whose **objects** are all sets and whose **morphisms** are all maps, and the **composite** of two morphisms $f : A \rightarrow B$ and $g : B \rightarrow C$, i.e., two maps, is the composition map $g \circ f : A \rightarrow C$, and for each object B in \mathcal{S} , the identity morphism $1_B : B \rightarrow B$ is just the identity map 1_B .

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$$(f_2 \circ f_1) \circ (g_1 \circ g_2)$$

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 $(f_2 \circ f_1) \circ (g_1 \circ g_2) = f_2 \circ (f_1 \circ (g_1 \circ g_2))$ **by the associativity**

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$$\begin{aligned}(f_2 \circ f_1) \circ (g_1 \circ g_2) &= f_2 \circ (f_1 \circ (g_1 \circ g_2)) \\ &= f_2 \circ ((f_1 \circ g_1) \circ g_2) \text{ by the associativity}\end{aligned}$$

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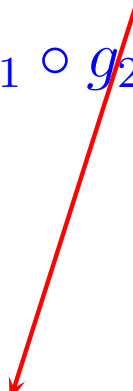
$$\begin{aligned}(f_2 \circ f_1) \circ (g_1 \circ g_2) &= f_2 \circ (f_1 \circ (g_1 \circ g_2)) \\ &= f_2 \circ ((f_1 \circ g_1) \circ g_2) \\ &= f_2 \circ (1_B \circ g_2) = f_2 \circ g_2\end{aligned}$$

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Proof. Assume $f_1 \circ g_1 = 1_B$, $g_1 \circ f_1 = 1_A$, and $f_2 \circ g_2 = 1_C$, $g_2 \circ f_2 = 1_B$. We want to show that $(f_2 \circ f_1) \circ (g_1 \circ g_2) = 1_C$ and $(g_1 \circ g_2) \circ (f_2 \circ f_1) = 1_A$.

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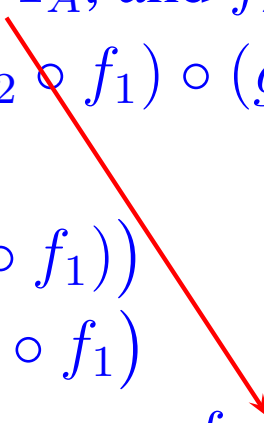
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More Examples

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- $\text{hom}(G, G)$ is the set of elements of G , i.e., $\text{hom}(G, G) = G$ as sets;
- the composite of two morphisms a, b is the product ab given by the binary operation in G ; note that the binary operation in G is associative;
- the identity element e of G is the identity morphism 1_G .

Note that in this category, **every morphism is an equivalence**, because every element of G has an inverse.

A Category of Morphisms

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- a morphism (α, β) of \mathcal{D} is an equivalence if and only if α and β are both equivalence of \mathcal{C} .