Modern Algebra I

Lecture 3

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Section I.5: Normality, Quotient Groups, and Homomorphisms

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Section I.6: Symmetric, Alternating, and Dihedral Groups

Chapter I

Section I.5: Normality, Quotient Groups, and Homomorphisms

Theorem (5.5)

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because f is a homomorphism

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$$f(gag^{-1}) = f(g)f(a)f(g)^{-1} = f(g)ef(g)^{-1} = e$$

and this implies $gag^{-1} \in \operatorname{Ker} f$.

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Proof. We first show that π is a homomorphism.

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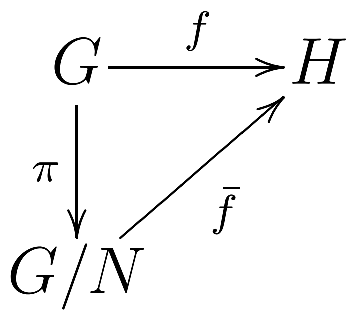

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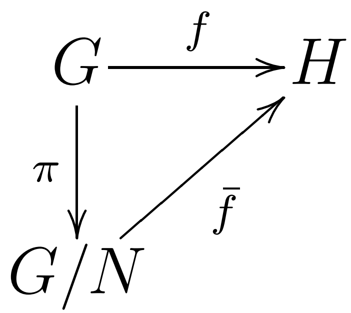

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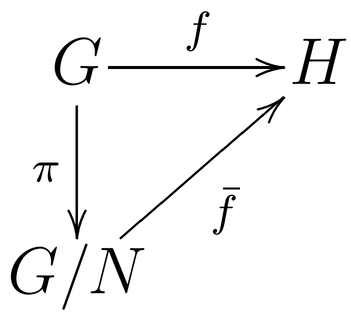

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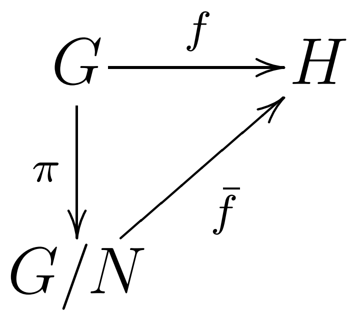

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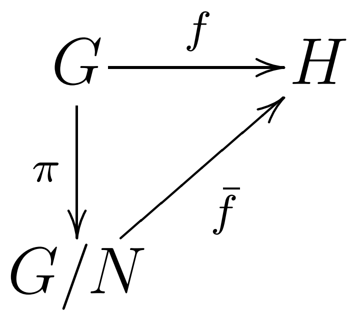

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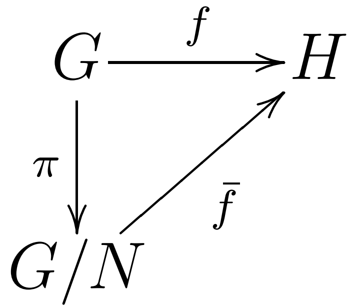

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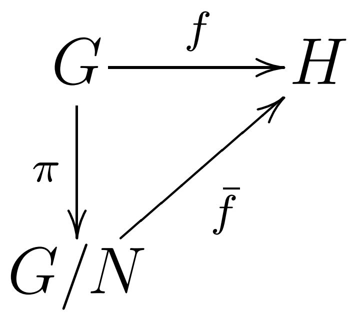

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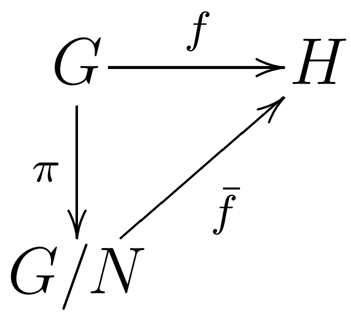

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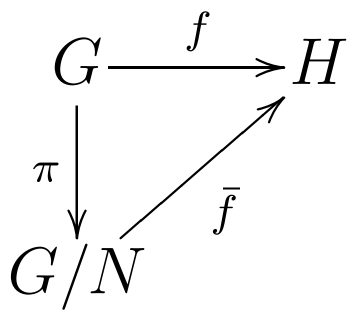

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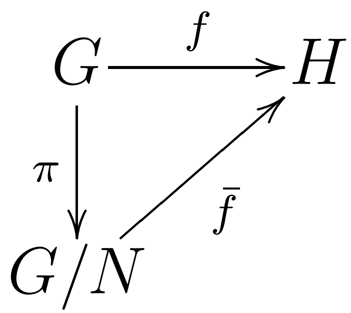

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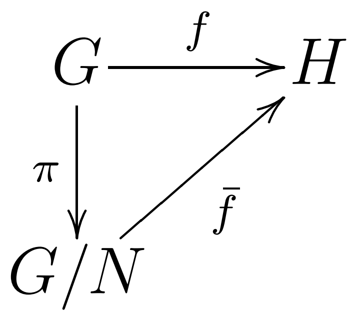

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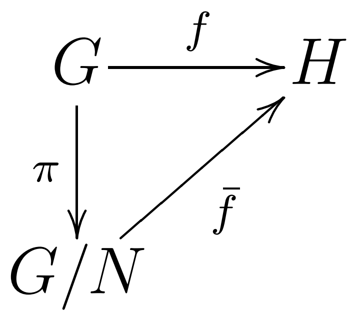

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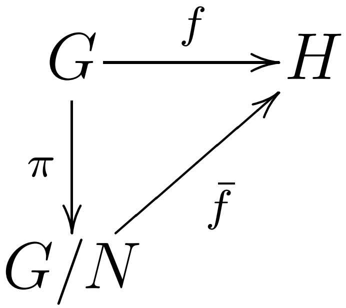

is commutative,
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Proof. We first show that $\bar{f}$ is well-defined. Note that
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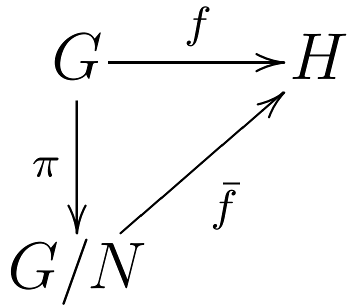

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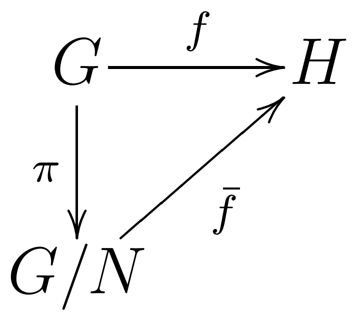

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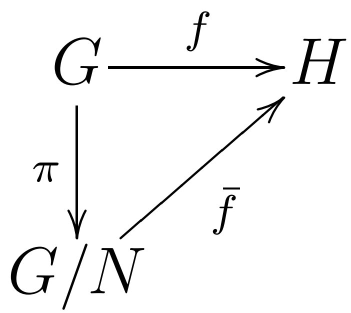

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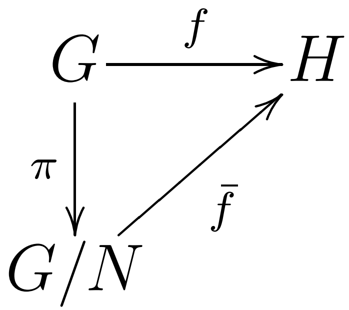

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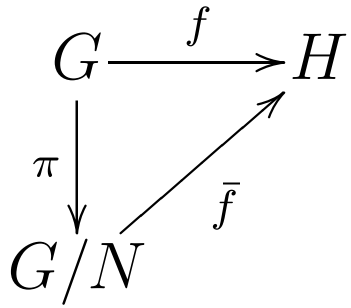

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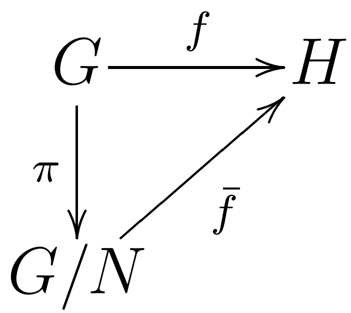

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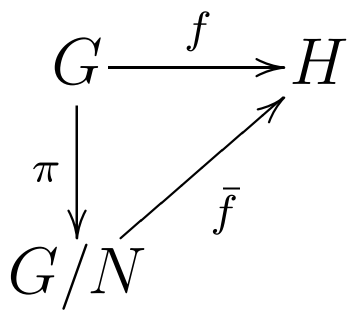

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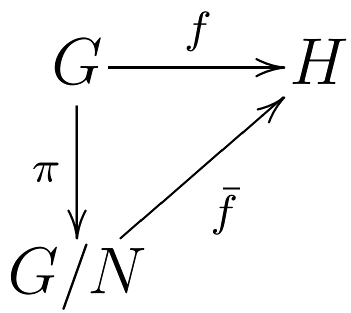

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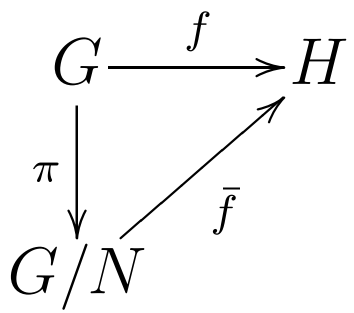

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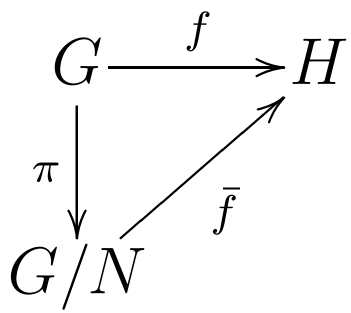

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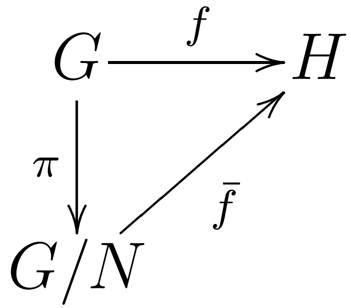

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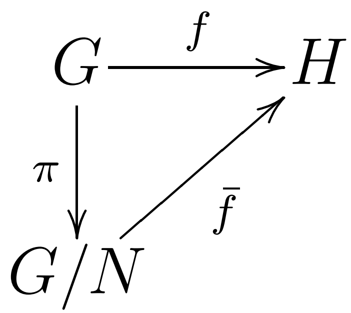

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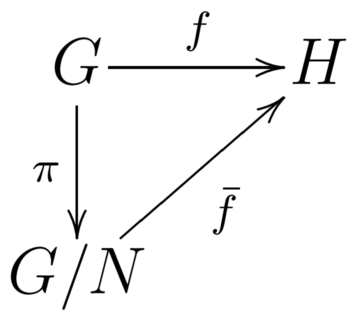

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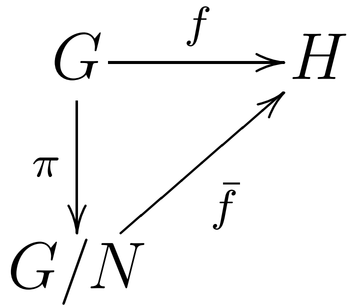

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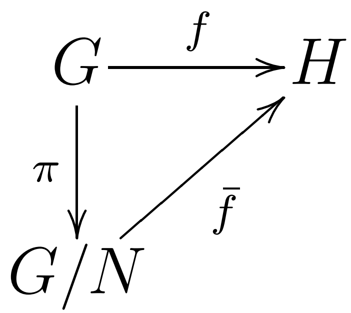

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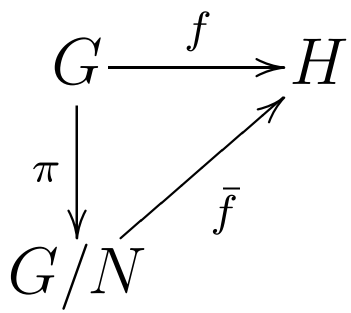

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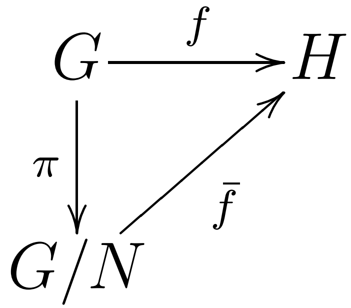

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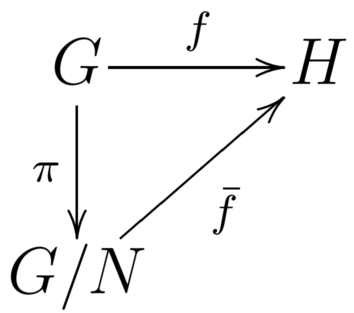

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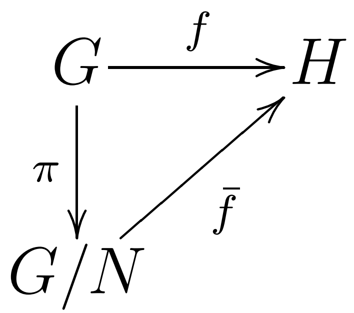

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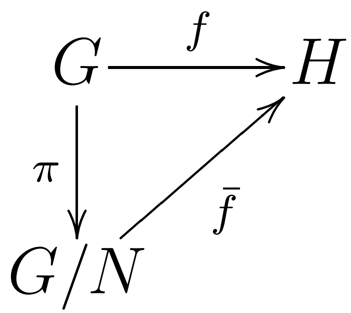

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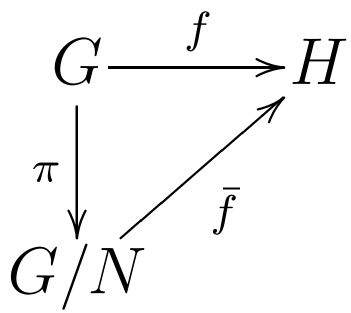

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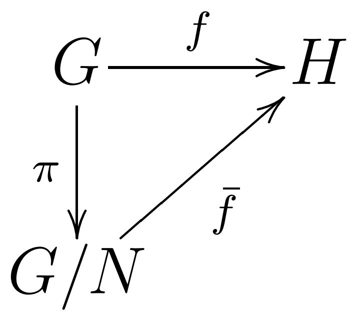

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First Isomorphism Theorem

Now, we use Theorem (5.6) to show the First Isomorphism Theorem.

First Isomorphism Theorem

Theorem (5.6)**.** Let $f : G \longrightarrow H$ be a homomorphism of groups and let N be a normal subgroup of G such that $N \subseteq \operatorname{Ker} f$.

First Isomorphism Theorem

Theorem (5.6)**.** Let $f : G \longrightarrow H$ be a homomorphism of groups and let N be a normal subgroup of G such that $N \subseteq \mathrm{Ker}\, f$. Then there is a unique homomorphism $\bar{f} : G/N \longrightarrow H$ such that $\bar{f}(aN) = f(a)$ for all $a \in G$.

First Isomorphism Theorem

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First Isomorphism Theorem

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First Isomorphism Theorem

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- $\operatorname{Im} \bar{f} = \operatorname{Im} f$ and $\operatorname{Ker} \bar{f} = (\operatorname{Ker} f)/N$;
- $\bar{f}$ is an isomorphism if and only if f is an epimorphism and $N = \operatorname{Ker} f$.

First Isomorphism Theorem

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Corollary (5.7)**. (First Isomorphism Theorem)** If $f : G \longrightarrow H$ is a homomorphism of groups,

First Isomorphism Theorem

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Corollary (5.7)**. (First Isomorphism Theorem)** If $f : G \longrightarrow H$ is a homomorphism of groups, then f induces an isomorphism

First Isomorphism Theorem

Theorem (5.6)**.** Let $f : G \to H$ be a homomorphism of groups and let N be a normal subgroup of G such that $N \subseteq \operatorname{Ker} f$. Then there is a unique homomorphism $\bar{f} : G/N \to H$ such that $\bar{f}(aN) = f(a)$ for all $a \in G$. Moreover,

- $\operatorname{Im} \bar{f} = \operatorname{Im} f$ and $\operatorname{Ker} \bar{f} = (\operatorname{Ker} f)/N$;
- $\bar{f}$ is an isomorphism if and only if f is an epimorphism and $N = \operatorname{Ker} f$.

Corollary (5.7)**. (First Isomorphism Theorem)** If $f : G \to H$ is a homomorphism of groups, then f induces an isomorphism $G/\operatorname{Ker} f \cong \operatorname{Im} f$.

First Isomorphism Theorem

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Corollary (5.7)**. (First Isomorphism Theorem)** If $f : G \longrightarrow H$ is a homomorphism of groups, then f induces an isomorphism $G/\operatorname{Ker} f \cong \operatorname{Im} f$.

Proof.

First Isomorphism Theorem

Theorem (5.6)**.** Let $f : G \longrightarrow H$ be a homomorphism of groups and let N be a normal subgroup of G such that $N \subseteq \operatorname{Ker} f$. Then there is a unique homomorphism $\bar{f} : G/N \longrightarrow H$ such that $\bar{f}(aN) = f(a)$ for all $a \in G$. Moreover,

- $\operatorname{Im} \bar{f} = \operatorname{Im} f$ and $\operatorname{Ker} \bar{f} = (\operatorname{Ker} f)/N$;
- $\bar{f}$ is an isomorphism if and only if f is an epimorphism and $N = \operatorname{Ker} f$.

Corollary (5.7)**.** **(First Isomorphism Theorem)** If $f : G \longrightarrow H$ is a homomorphism of groups, then f induces an isomorphism $G/\operatorname{Ker} f \cong \operatorname{Im} f$.

Proof. Note that $f : G \rightarrow \operatorname{Im} f$ is an epimorphism.

First Isomorphism Theorem

Theorem (5.6)**.** Let $f : G \longrightarrow H$ be a homomorphism of groups and let N be a normal subgroup of G such that $N \subseteq \operatorname{Ker} f$. Then there is a unique homomorphism $\bar{f} : G/N \longrightarrow H$ such that $\bar{f}(aN) = f(a)$ for all $a \in G$. Moreover,

- $\operatorname{Im} \bar{f} = \operatorname{Im} f$ and $\operatorname{Ker} \bar{f} = (\operatorname{Ker} f)/N$;
- $\bar{f}$ is an isomorphism if and only if f is an epimorphism and $N = \operatorname{Ker} f$.

Corollary (5.7)**. (First Isomorphism Theorem)** If $f : G \longrightarrow H$ is a homomorphism of groups, then f induces an isomorphism $G/\operatorname{Ker} f \cong \operatorname{Im} f$.

Proof. Note that $f : G \to \operatorname{Im} f$ is an epimorphism. Apply Theorem (5.6) with $N = \operatorname{Ker} f$.

First Isomorphism Theorem

Theorem (5.6)**.** Let $f : G \longrightarrow H$ be a homomorphism of groups and let N be a normal subgroup of G such that $N \subseteq \operatorname{Ker} f$. Then there is a unique homomorphism $\bar{f} : G/N \longrightarrow H$ such that $\bar{f}(aN) = f(a)$ for all $a \in G$. Moreover,

- $\operatorname{Im} \bar{f} = \operatorname{Im} f$ and $\operatorname{Ker} \bar{f} = (\operatorname{Ker} f)/N$;
- $\bar{f}$ is an isomorphism if and only if f is an epimorphism and $N = \operatorname{Ker} f$.

Corollary (5.7)**.** **(First Isomorphism Theorem)** If $f : G \longrightarrow H$ is a homomorphism of groups, then f induces an isomorphism $G/\operatorname{Ker} f \cong \operatorname{Im} f$.

Proof. Note that $f : G \longrightarrow \operatorname{Im} f$ is an epimorphism. Apply Theorem (5.6) with $N = \operatorname{Ker} f$. Then we get that $\bar{f} : G/\operatorname{Ker} f \longrightarrow \operatorname{Im} f$ is an isomorphism.

First Isomorphism Theorem

Theorem (5.6)**.** Let $f : G \longrightarrow H$ be a homomorphism of groups and let N be a normal subgroup of G such that $N \subseteq \operatorname{Ker} f$. Then there is a unique homomorphism $\bar{f} : G/N \longrightarrow H$ such that $\bar{f}(aN) = f(a)$ for all $a \in G$. Moreover,

- $\operatorname{Im} \bar{f} = \operatorname{Im} f$ and $\operatorname{Ker} \bar{f} = (\operatorname{Ker} f)/N$;
- $\bar{f}$ is an isomorphism if and only if f is an epimorphism and $N = \operatorname{Ker} f$.

Corollary (5.7)**. (First Isomorphism Theorem)** If $f : G \longrightarrow H$ is a homomorphism of groups, then f induces an isomorphism $G/\operatorname{Ker} f \cong \operatorname{Im} f$. In particular,

First Isomorphism Theorem

Theorem (5.6)**.** Let $f : G \longrightarrow H$ be a homomorphism of groups and let N be a normal subgroup of G such that $N \subseteq \operatorname{Ker} f$. Then there is a unique homomorphism $\bar{f} : G/N \longrightarrow H$ such that $\bar{f}(aN) = f(a)$ for all $a \in G$. Moreover,

- $\operatorname{Im} \bar{f} = \operatorname{Im} f$ and $\operatorname{Ker} \bar{f} = (\operatorname{Ker} f)/N$;
- $\bar{f}$ is an isomorphism if and only if f is an epimorphism and $N = \operatorname{Ker} f$.

Corollary (5.7)**. (First Isomorphism Theorem)** If $f : G \longrightarrow H$ is a homomorphism of groups, then f induces an isomorphism $G/\operatorname{Ker} f \cong \operatorname{Im} f$. In particular, if f is an epimorphism,

First Isomorphism Theorem

Theorem (5.6). Let $f : G \longrightarrow H$ be a homomorphism of groups and let N be a normal subgroup of G such that $N \subseteq \operatorname{Ker} f$. Then there is a unique homomorphism $\bar{f} : G/N \longrightarrow H$ such that $\bar{f}(aN) = f(a)$ for all $a \in G$. Moreover,

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Corollary (5.7). **(First Isomorphism Theorem)** If $f : G \longrightarrow H$ is a homomorphism of groups, then f induces an isomorphism $G/\operatorname{Ker} f \cong \operatorname{Im} f$. In particular, if f is an epimorphism, then $G/\operatorname{Ker} f \cong H$.

Second Isomorphism Theorem

We use the First Isomorphism Theorem to prove the Second Isomorphism Theorem.

Second Isomorphism Theorem

Corollary (5.7)**.** **(First Isomorphism Theorem)**

Second Isomorphism Theorem

Corollary (5.7)**. (First Isomorphism Theorem)** If $f : G \to H$ is an epimorphism,

Second Isomorphism Theorem

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Second Isomorphism Theorem

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Corollary (5.9)**. (Second Isomorphism Theorem)**

Second Isomorphism Theorem

Corollary (5.7)**. (First Isomorphism Theorem)** If $f : G \to H$ is an epimorphism, then $G/\operatorname{Ker} f \cong H$.

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Second Isomorphism Theorem

Corollary (5.7)**. (First Isomorphism Theorem)** If $f : G \to H$ is an epimorphism, then $G/\operatorname{Ker} f \cong H$.

Corollary (5.9)**. (Second Isomorphism Theorem)** If K and N are subgroups of a group G, with N normal in G,

Second Isomorphism Theorem

Corollary (5.7)**. (First Isomorphism Theorem)** If $f : G \to H$ is an epimorphism, then $G/\operatorname{Ker} f \cong H$.

Corollary (5.9)**. (Second Isomorphism Theorem)** If K and N are subgroups of a group G, with N normal in G, then $K/(N \cap K) \cong NK/N$.

Second Isomorphism Theorem

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Corollary (5.9)**.** **(Second Isomorphism Theorem)** If K and N are subgroups of a group G, with N normal in G, then $K/(N \cap K) \cong NK/N$.

Proof. Since $N \lhd G$,

Second Isomorphism Theorem

Corollary (5.7)**. (First Isomorphism Theorem)** If $f : G \to H$ is an epimorphism, then $G/\operatorname{Ker} f \cong H$.

Corollary (5.9)**. (Second Isomorphism Theorem)** If K and N are subgroups of a group G, with N normal in G, then $K/(N \cap K) \cong NK/N$.

Proof. Since $N \triangleleft G$, NK is a subgroup of G

Second Isomorphism Theorem

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Proof. Since $N \lhd G$, NK is a subgroup of G and so $N \lhd NK$.

Second Isomorphism Theorem

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Corollary (5.9)**. (Second Isomorphism Theorem)** If K and N are subgroups of a group G, with N normal in G, then $K/(N \cap K) \cong NK/N$.

Proof. Since $N \lhd G$, NK is a subgroup of G and so $N \lhd NK$. Consider the map $f : K \to NK/N$ defined by $k \mapsto kN$.

Second Isomorphism Theorem

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Consider the map $f : K \to NK/N$ defined by $k \mapsto kN$.
Since $f(k_1 k_2)$

Second Isomorphism Theorem

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Proof. Since $N \lhd G$, NK is a subgroup of G and so $N \lhd NK$.
Consider the map $f : K \to NK/N$ defined by $k \mapsto kN$.
Since $f(k_1k_2) = k_1k_2N$

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Proof. Since $N \lhd G$, NK is a subgroup of G and so $N \lhd NK$. Consider the map $f : K \to NK/N$ defined by $k \mapsto kN$. Since $f(k_1k_2) = k_1k_2N = (k_1N)(k_2N)$

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Proof. Since $N \triangleleft G$, NK is a subgroup of G and so $N \triangleleft NK$.
Consider the map $f : K \to NK/N$ defined by $k \mapsto kN$.
Since $f(k_1k_2) = k_1k_2N = (k_1N)(k_2N) = f(k_1)f(k_2)$,

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Proof. Since $N \lhd G$, NK is a subgroup of G and so $N \lhd NK$. Consider the map $f : K \to NK/N$ defined by $k \mapsto kN$. Since $f(k_1k_2) = k_1k_2N = (k_1N)(k_2N) = f(k_1)f(k_2)$, f is a homomorphism.

Second Isomorphism Theorem

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Proof. Since $N \lhd G$, NK is a subgroup of G and so $N \lhd NK$. Consider the map $f : K \to NK/N$ defined by $k \mapsto kN$. Since $f(k_1k_2) = k_1k_2N = (k_1N)(k_2N) = f(k_1)f(k_2)$, f is a homomorphism. Moreover,

$k \in \operatorname{Ker} f$

Second Isomorphism Theorem

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Proof. Since $N \lhd G$, NK is a subgroup of G and so $N \lhd NK$. Consider the map $f : K \to NK/N$ defined by $k \mapsto kN$. Since $f(k_1k_2) = k_1k_2N = (k_1N)(k_2N) = f(k_1)f(k_2)$, f is a homomorphism. Moreover,

$k \in \operatorname{Ker} f \iff kN = N$

Second Isomorphism Theorem

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Proof. Since $N \lhd G$, NK is a subgroup of G and so $N \lhd NK$. Consider the map $f : K \to NK/N$ defined by $k \mapsto kN$. Since $f(k_1k_2) = k_1k_2N = (k_1N)(k_2N) = f(k_1)f(k_2)$, f is a homomorphism. Moreover,

$k \in \operatorname{Ker} f \iff kN = N \iff k \in N$,

Second Isomorphism Theorem

Corollary (5.7)**. (First Isomorphism Theorem)** If $f : G \to H$ is an epimorphism, then $G/\operatorname{Ker} f \cong H$.

Corollary (5.9)**. (Second Isomorphism Theorem)** If K and N are subgroups of a group G, with N normal in G, then $K/(N \cap K) \cong NK/N$.

Proof. Since $N \lhd G$, NK is a subgroup of G and so $N \lhd NK$. Consider the map $f : K \to NK/N$ defined by $k \mapsto kN$. Since $f(k_1k_2) = k_1k_2N = (k_1N)(k_2N) = f(k_1)f(k_2)$, f is a homomorphism. Moreover,

$k \in \operatorname{Ker} f \iff kN = N \iff k \in N$, so $\operatorname{Ker} f = N \cap K$.

Second Isomorphism Theorem

Corollary (5.7). **(First Isomorphism Theorem)** If $f : G \to H$ is an epimorphism, then $G/\operatorname{Ker} f \cong H$.

Corollary (5.9). **(Second Isomorphism Theorem)** If K and N are subgroups of a group G, with N normal in G, then $K/(N \cap K) \cong NK/N$.

Proof. Since $N \lhd G$, NK is a subgroup of G and so $N \lhd NK$. Consider the map $f : K \to NK/N$ defined by $k \mapsto kN$. Since $f(k_1k_2) = k_1k_2N = (k_1N)(k_2N) = f(k_1)f(k_2)$, f is a homomorphism. Moreover, $k \in \operatorname{Ker} f \iff kN = N \iff k \in N$, so $\operatorname{Ker} f = N \cap K$. On the other hand, for all $nk \in NK$ with $n \in N$ and $k \in K$,

Second Isomorphism Theorem

Corollary (5.7)**. (First Isomorphism Theorem)** If $f : G \to H$ is an epimorphism, then $G/\operatorname{Ker} f \cong H$.

Corollary (5.9)**. (Second Isomorphism Theorem)** If K and N are subgroups of a group G, with N normal in G, then $K/(N \cap K) \cong NK/N$.

Proof. Since $N \lhd G$, NK is a subgroup of G and so $N \lhd NK$.
Consider the map $f : K \to NK/N$ defined by $k \mapsto kN$.
Since $f(k_1k_2) = k_1k_2N = (k_1N)(k_2N) = f(k_1)f(k_2)$, f is a homomorphism. Moreover,
$k \in \operatorname{Ker} f \iff kN = N \iff k \in N$, so $\operatorname{Ker} f = N \cap K$.
On the other hand, for all $nk \in NK$ with $n \in N$ and $k \in K$,
$nkN = kN$ since $k^{-1}(nk) = k^{-1}nk \in N$.

Second Isomorphism Theorem

Corollary (5.7)**. (First Isomorphism Theorem)** If $f : G \to H$ is an epimorphism, then $G/\operatorname{Ker} f \cong H$.

Corollary (5.9)**. (Second Isomorphism Theorem)** If K and N are subgroups of a group G, with N normal in G, then $K/(N \cap K) \cong NK/N$.

Proof. Since $N \lhd G$, NK is a subgroup of G and so $N \lhd NK$. Consider the map $f : K \to NK/N$ defined by $k \mapsto kN$. Since $f(k_1k_2) = k_1k_2N = (k_1N)(k_2N) = f(k_1)f(k_2)$, f is a homomorphism. Moreover, $k \in \operatorname{Ker} f \iff kN = N \iff k \in N$, so $\operatorname{Ker} f = N \cap K$. On the other hand, for all $nk \in NK$ with $n \in N$ and $k \in K$, $nkN = kN$ since $k^{-1}(nk) = k^{-1}nk \in N$. Therefore, f is an epimorphism

Second Isomorphism Theorem

Corollary (5.7)**. (First Isomorphism Theorem)** If $f : G \to H$ is an epimorphism, then $G/\operatorname{Ker} f \cong H$.

Corollary (5.9)**. (Second Isomorphism Theorem)** If K and N are subgroups of a group G, with N normal in G, then $K/(N \cap K) \cong NK/N$.

Proof. Since $N \lhd G$, NK is a subgroup of G and so $N \lhd NK$. Consider the map $f : K \to NK/N$ defined by $k \mapsto kN$. Since $f(k_1k_2) = k_1k_2N = (k_1N)(k_2N) = f(k_1)f(k_2)$, f is a homomorphism. Moreover,
$k \in \operatorname{Ker} f \iff kN = N \iff k \in N$, so $\operatorname{Ker} f = N \cap K$.
On the other hand, for all $nk \in NK$ with $n \in N$ and $k \in K$, $nkN = kN$ since $k^{-1}(nk) = k^{-1}nk \in N$. Therefore, f is an epimorphism with $\operatorname{Ker} f = N \cap K$.

Second Isomorphism Theorem

Corollary (5.7). **(First Isomorphism Theorem)** If $f : G \to H$ is an epimorphism, then $G/\operatorname{Ker} f \cong H$.

Corollary (5.9). **(Second Isomorphism Theorem)** If K and N are subgroups of a group G, with N normal in G, then $K/(N \cap K) \cong NK/N$.

Proof. Since $N \lhd G$, NK is a subgroup of G and so $N \lhd NK$. Consider the map $f : K \to NK/N$ defined by $k \mapsto kN$. Since $f(k_1k_2) = k_1k_2N = (k_1N)(k_2N) = f(k_1)f(k_2)$, f is a homomorphism. Moreover,

$k \in \operatorname{Ker} f \iff kN = N \iff k \in N$, so $\operatorname{Ker} f = N \cap K$.

On the other hand, for all $nk \in NK$ with $n \in N$ and $k \in K$, $nkN = kN$ since $k^{-1}(nk) = k^{-1}nk \in N$. Therefore, f is an epimorphism with $\operatorname{Ker} f = N \cap K$. By the First Isomorphism Theorem,

Second Isomorphism Theorem

Corollary (5.7)**.** **(First Isomorphism Theorem)** If $f : G \to H$ is an epimorphism, then $G/\operatorname{Ker} f \cong H$.

Corollary (5.9)**.** **(Second Isomorphism Theorem)** If K and N are subgroups of a group G, with N normal in G, then $K/(N \cap K) \cong NK/N$.

Proof. Since $N \lhd G$, NK is a subgroup of G and so $N \lhd NK$. Consider the map $f : K \to NK/N$ defined by $k \mapsto kN$. Since $f(k_1k_2) = k_1k_2N = (k_1N)(k_2N) = f(k_1)f(k_2)$, f is a homomorphism. Moreover,
$k \in \operatorname{Ker} f \iff kN = N \iff k \in N$, so $\operatorname{Ker} f = N \cap K$.
On the other hand, for all $nk \in NK$ with $n \in N$ and $k \in K$, $nkN = kN$ since $k^{-1}(nk) = k^{-1}nk \in N$. Therefore, f is an epimorphism with $\operatorname{Ker} f = N \cap K$. By the First Isomorphism Theorem, $K/(N \cap K) \cong NK/N$.

Third Isomorphism Theorem

We use the First Isomorphism Theorem to prove the Third Isomorphism Theorem.

Third Isomorphism Theorem

Corollary (5.7)**.** **(First Isomorphism Theorem)** If $f : G \to H$ is an epimorphism, then $G/\operatorname{Ker} f \cong H$.

Third Isomorphism Theorem

Corollary (5.7)**. (First Isomorphism Theorem)** If $f : G \to H$ is an epimorphism, then $G/\operatorname{Ker} f \cong H$.

Corollary (5.9)**. (Third Isomorphism Theorem)**

Third Isomorphism Theorem

Corollary (5.7)**. (First Isomorphism Theorem)** If $f : G \to H$ is an epimorphism, then $G/\operatorname{Ker} f \cong H$.

Corollary (5.9)**. (Third Isomorphism Theorem)** If H and K are normal subgroups of a group G

Third Isomorphism Theorem

Corollary (5.7)**. (First Isomorphism Theorem)** If $f : G \to H$ is an epimorphism, then $G/\operatorname{Ker} f \cong H$.

Corollary (5.9)**. (Third Isomorphism Theorem)** If H and K are normal subgroups of a group G such that $K \leq H$,

Third Isomorphism Theorem

Corollary (5.7)**. (First Isomorphism Theorem)** If $f : G \to H$ is an epimorphism, then $G/\operatorname{Ker} f \cong H$.

Corollary (5.9)**. (Third Isomorphism Theorem)** If H and K are normal subgroups of a group G such that $K \leq H$, then H/K is a normal subgroup of G/K

Third Isomorphism Theorem

Corollary (5.7)**.** **(First Isomorphism Theorem)** If $f : G \to H$ is an epimorphism, then $G/\operatorname{Ker} f \cong H$.

Corollary (5.9)**.** **(Third Isomorphism Theorem)** If H and K are normal subgroups of a group G such that $K \leq H$, then H/K is a normal subgroup of G/K and $(G/K)/(H/K) \cong G/H$.

Third Isomorphism Theorem

Corollary (5.7)**.** **(First Isomorphism Theorem)** If $f : G \to H$ is an epimorphism, then $G/\operatorname{Ker} f \cong H$.

Corollary (5.9)**.** **(Third Isomorphism Theorem)** If H and K are normal subgroups of a group G such that $K \leq H$, then H/K is a normal subgroup of G/K and $(G/K)/(H/K) \cong G/H$.

Proof. Consider the canonical epimorphism $\pi : G \to G/H$ defined by $a \mapsto aH$.

Third Isomorphism Theorem

Corollary (5.7)**. (First Isomorphism Theorem)** If $f : G \to H$ is an epimorphism, then $G/\operatorname{Ker} f \cong H$.

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Proof. From the previous Remark, we know that ϕ is well-defined and that $K \lhd G \implies f(K) \lhd H$. Hence it remains to show that ϕ is bijective and $K \lhd G \Longleftarrow f(K) \lhd H$.

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If $f : G \to H$ is an epimorphism of groups, then the assignment $K \mapsto f(K)$ defines a one-to-one correspondence between $S_f(G) = \{K \mid \operatorname{Ker} f \le K \le G\}$ and $S(H) = \{M \mid M \le H\}$; in other words, the map $\phi : S_f(G) \to S(H)$ defined by $\phi(K) = f(K)$ is bijective. Moreover, under this correspondence, normal subgroups correspond to normal subgroups, i.e., $K \lhd G \iff f(K) \lhd H$.

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Next we will use Theorem (5.11) to investigate the subgroups and normal subgroups of a quotient group G/N.

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Remark. $n!$ is read "n factorial".

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- A 2-cycle is called a transposition.

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Proof. We need to show that $\sigma\tau(k) = \tau\sigma(k) \ \ \forall k \in I_n$.

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Hence $H \lhd G$.

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We are finally ready to prove Theorem (6.8).