

Modern Algebra I

Lecture 16

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Section III.2: Ideals

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Remark. Left proper ideals and right proper ideals are defined similarly as in the above **Definition**. The same statement as in the above **Remark** holds true for both left ideals and right ideals.

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so we have $ra + as + \sum_{i=1}^m r_i a s_i = ra + sa + \sum_{i=1}^m r_i s_i a = (r + s + \sum_{i=1}^m r_i s_i)a$, which is an element of the form ra .

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Definition. The map π is called the **canonical epimorphism** or the **canonical projection**.

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Indeed, let $a + I, b + I \in J/I$ with $a, b \in J$ and let $r + I \in R/I$ with $r \in R$. Then

- $(a + I) - (b + I) = (a - b) + I \in J/I$, because $a - b \in J$;
- $(r + I)(a + I) = ra + I \in J/I$ and
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First, because π is onto, $\pi(\pi^{-1}(\mathcal{J})) = \mathcal{J}$. Secondly, we always have $\pi^{-1}(\pi(J)) \supseteq J$. Conversely, let $a \in \pi^{-1}(\pi(J))$, i.e., $\pi(a) \in \pi(J)$, i.e., $a + I \in J/I$, i.e., $a + I = b + I$ for some $b \in J$. Then $a - b \in I \subseteq J$. Since $a = (a - b) + b$, $a \in J$ and this completes the proof.

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Corollary. In a commutative ring R , an ideal P with $P \neq R$ is prime if and only if for $a, b \in R$, $ab \in P \implies a \in P$ or $b \in P$.

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Thus, $AB \not\subseteq P$.

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Proof. Suppose there exists a maximal ideal M which is not a prime ideal. Then $\exists a, b \in R$ such that $ab \in M$ but $a \notin M$ and $b \notin M$. Then $M \subsetneq M + (a)$ and $M \subsetneq M + (b)$. Since M is maximal, $M + (a) = M + (b) = R$. Then

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Corollary. If R is a commutative ring with $1_R \neq 0$, then an ideal M in R is maximal if and only if R/M is a field.

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Thus, M is a maximal ideal.

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Let R be a ring with $1_R \neq 0$ and let M be an ideal in R .

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Happy New Year!!