Modern Algebra I

Lecture 15

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- $(\sum_{i=1}^{n} a_i)\left(\sum_{j=1}^{m} b_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_i b_j, \forall a_i, b_j \in R$.

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$$a_L^{-1} = a_L^{-1}1_R = a_L^{-1}(aa_R^{-1}) = (a_L^{-1}a)a_R^{-1} = 1_R a_R^{-1} = a_R^{-1}.$$

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Remark. A field is a commutative ring with an identity $1_R \neq 0$ such that every nonzero element is a unit.

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This is because if every nonzero element of R is a unit, then R has no zero divisors. More precisely, suppose $a \in R \setminus \{0\}$. Then $a^{-1} \in R$. If $ab = 0$ for some $b \in R$, then $a^{-1}ab = a^{-1}0 \Rightarrow b = 0$; if $ba = 0$ for some $b \in R$,

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With this addition and composition of functions as multiplication, $\mathrm{End}A$ is a (usually noncommutative) ring with identity $1_A : A \to A$. These two are clear.

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by the definition of $(f+g)+h$

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so $(f+g)+h = f+(g+h)$.

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because f is a homomorphism

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Hence $-f$ is a group homomorphism and so $-f \in \mathrm{End}A$.

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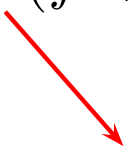

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Remark. The class of all rings together with all ring homomorphisms forms a (concrete) category.

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Every ring R may be embedded in a ring S with identity. Moreover, the ring S (which is not unique) may be chosen to be either of characteristic zero or of the same characteristic as R.

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- Take $S = \mathbb{Z}_n \oplus R$ and $\forall \overline{k_1}, \overline{k_2} \in \mathbb{Z}$ and $\forall r_1, r_2 \in R$, let

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Moreover, $f : R \longrightarrow S$ is a monomorphism,

$$r \longmapsto (\overline{0}, r)$$

i.e., an embedding.

Exercise for Section III.1

2, 3, 5, 6, 11, 12, 14, 16.

Chapter III: Rings

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Section III.2: Ideals

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Example. Consider the ring homomorphism $f : \mathbb{Z} \to \mathbb{Q}$ given by $f(n) = n$. Then $\mathrm{Im}\ f = \mathbb{Z}$, which is not an ideal of $\mathbb{Q}$; for example, $\frac{1}{2} \cdot 3 = \frac{3}{2} \notin \mathbb{Z}$, even though $\frac{1}{2} \in \mathbb{Q}$ and $3 \in \mathbb{Z}$. In fact, the only ideal I of $\mathbb{Z}$ such that $f(I)$ is an ideal of $\mathbb{Q}$

Remark

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Proof. It is clear that (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii).

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Proof. It is clear that (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii). Hence we only need to show (iii) $\Longrightarrow$ (i). Let $u \in I$ be a unit in R. Then there exists $u^{-1} \in R$. For all $r \in R$, since $r = (ru^{-1})u$, $ru^{-1} \in R$, and $u \in I$, we have $r = (ru^{-1})u \in I$. Thus $R \subseteq I$ and so $I = R$.

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Proof. Since (i) tells us that I is a subgroup of R, this follows from our remark to the definition of ideals.

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  $= \{a_1 + a_2 + \cdots + a_n \mid a_i \in A_i,\ \forall i = 1, 2, \ldots, n\}$.
- $AB = \{a_1 b_1 + \cdots + a_m b_m \mid m \in \mathbb{N},\ a_i \in A,\ b_i \in B\}$.
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$$(a+I)(b+I) = ab+I \quad \forall a, b \in R.$$

Moreover, if R is commutative or has identity, then the same is true for R/I.

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Proof. If R is commutative, it is easy to see that R/I is also commutative. If R is a ring with identity 1_R, it is easy to see that $1_R + I$ is the identity of R/I.

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Proof. In a previous remark, we have proved that $\mathrm{Ker}\ f$ is an ideal in R. Conversely, since I is a subgroup of the abelian group R, by Theorem (I.5.5), the map π is an epimorphism of groups with kernel I. Moreover, for all $a, b \in R$, $\pi(ab) = ab + I$ $= (a+I)(b+I) = \pi(a)\pi(b)$. Thus π is a ring homomorphism, and so π is an epimorphism of rings.

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Definition. The map π is called the canonical epimorphism or the canonical projection.

Exercise for Section III.2

1, 2, 3, 10, 11, 13, 16, 18.