Modern Algebra I

Lecture 14

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2009, Fall

Chapter II

The Structure of Groups

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Section II.6: Classification of Finite Groups

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- if $|G| = p$ is a prime, then $G \cong \mathbb{Z}_p$;
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and got

Order	*Distinct Groups*
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
12	

Order	*Distinct Groups*
13	
14	
15	

Order	*Distinct Groups*
1	$\langle e \rangle$
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
12	

Order	*Distinct Groups*
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14	
15	

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1	$\langle e \rangle$
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3	
4	
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6	
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8	
9	
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11	
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15	

Order	*Distinct Groups*	*Order*	*Distinct Groups*
1	$\langle e \rangle$	13	
2	$\mathbb{Z}_2$	14	
3	$\mathbb{Z}_3$	15	
4			
5	$\mathbb{Z}_5$		
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8			
9			
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- Q is a normal subgroup of G.
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- Moreover, $|PQ| = \frac{|P||Q|}{|P \cap Q|} = pq = |G|$ and so $PQ = G$.

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- $PQ = G$.

Hence $G \cong P \times Q$

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- Q is a normal subgroup of G.
- P is a normal subgroup of G.
- $P \cap Q = \{e\}$.
- $PQ = G$.

Hence $G \cong P \times Q \cong \mathbb{Z}_p \times \mathbb{Z}_q$

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- Q is a normal subgroup of G.
- P is a normal subgroup of G.
- $P \cap Q = \{e\}$.
- $PQ = G$.

Hence $G \cong P \times Q \cong \mathbb{Z}_p \times \mathbb{Z}_q \cong \mathbb{Z}_{pq}$ is cyclic.

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Lemma. Let G be a group of order pq, where $p < q$ are primes with $p \nmid q - 1$. Then $G \cong \mathbb{Z}_{pq}$.

Since $15 = 3 \cdot 5$ and since $3 < 5$ are primes that satisfy the condition in the above lemma,

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Order $2p$ with Odd Prime p

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$$= \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & i & \cdots & n-1 & n \\ 1 & n & n-1 & n-2 & \cdots & n+2-i & \cdots & 3 & 2 \end{pmatrix}.$$

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Theorem (I.6.13)**.** For each $n \ge 3$, the dihedral group D_n is a group of order $2n$

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Theorem (I.6.13)**.** For each $n \ge 3$, the dihedral group D_n is a group of order $2n$ whose generators a and b satisfy

(i) $a^n = e$;

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Therefore, we have

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3	$\mathbb{Z}_3$
4	$\mathbb{Z}_2 \oplus \mathbb{Z}_2,\ \mathbb{Z}_4$
5	$\mathbb{Z}_5$
6	$\mathbb{Z}_6,\ D_3 = S_3$
7	$\mathbb{Z}_7$
8	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2, \mathbb{Z}_2 \oplus \mathbb{Z}_4,\ \mathbb{Z}_8,\ Q_8,\ D_4$
9	$\mathbb{Z}_3 \oplus \mathbb{Z}_3,\ \mathbb{Z}_9$
10	$\mathbb{Z}_{10},\ D_5$
11	$\mathbb{Z}_{11}$
12	

Order	*Distinct Groups*
13	$\mathbb{Z}_{13}$
14	$\mathbb{Z}_{14},\ D_7$
15	$\mathbb{Z}_{15}$

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- Next, we will show in Proposition (6.4) that if G is a nonabelian group, then $G \cong A_4$, D_6, or T.

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$\star$ If $\operatorname{Ker}\phi = P$, then $P \lhd G$ and so P is the unique Sylow 3-subgroup of G. Therefore, G contains only two elements of order 3.

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Proof. Let G be a nonabelian group of order 12. Let P be a Sylow 3-subgroup of G. Then $|P| = 3$ and $[G : P] = 4$. By Proposition (4.8), there exists a homomorphism $\phi : G \to S_4$ such that $\operatorname{Ker}\phi \subseteq P$. Since $|P| = 3$, $\operatorname{Ker}\phi = \{e\}$ or $\operatorname{Ker}\phi = P$.

$\star$ If $\operatorname{Ker}\phi = \{e\}$, then ϕ is a monomorphism and so $G \cong \phi(G)$. Note that $\phi(G)$ is a subgroup of S_4 with $[S_4 : \phi(G)] = 2$. By Theorem (I.6.8), A_4 is the unique subgroup of S_4 of index 2, so $\phi(G) = A_4$ and so $G \cong A_4$.

$\star$ If $\operatorname{Ker}\phi = P$, then $P \lhd G$ and so P is the unique Sylow 3-subgroup of G. Therefore, G contains only two elements of order 3. Let c and $c^2 = c^{-1}$ be these two elements.

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- $cd = dc$, *since* $d \in C_G(c)$
- $|c| = 3$, $|d| = 2$, and $gcd(3, 2) = 1$.

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Hence $b^2 = e$, a, a^2, a^3, a^4, or a^5.

- If $b^2 = e$, then we have $G = \langle a, b \rangle$, $a^6 = b^2 = e$, $a^k \neq e$ for $0 < k < 6$, and $bab^{-1} = a^{-1}$.

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1	$\langle e \rangle$
2	$\mathbb{Z}_2$
3	$\mathbb{Z}_3$
4	$\mathbb{Z}_2 \oplus \mathbb{Z}_2,\ \mathbb{Z}_4$
5	$\mathbb{Z}_5$
6	$\mathbb{Z}_6,\ D_3 = S_3$
7	$\mathbb{Z}_7$
8	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2, \mathbb{Z}_2 \oplus \mathbb{Z}_4,\ \mathbb{Z}_8,\ Q_8,\ D_4$
9	$\mathbb{Z}_3 \oplus \mathbb{Z}_3,\ \mathbb{Z}_9$
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Proof. For the second case $q \mid p - 1$, we only need to show that if G is a nonabelian group of order pq, then G is isomorphic to the group K that is described above. The existence of such a group K are proved in Exercise 1 and 2.

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Let $p > q$ be prime numbers such that $q \mid p - 1$ and let G be a nonabelian group of order pq. By Cauchy's Theorem, there exist elements $a, b \in G$ such that $|a| = p$ and $|b| = q$.

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We want to show that if G is a nonabelian group of order pq, then G is isomorphic to K.

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Exercise for Section II.6

1, 2, 3, 4, 6, 7

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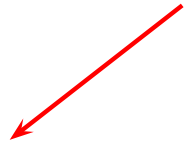

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Proof. The proofs for the last two statements are straight forward using the left and right distribution laws.

Zero Divisors

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Remark. A field is a commutative ring with an identity $1_R \neq 0$ such that every nonzero element is a unit.