

Modern Algebra I

Lecture 13

Jung-Chen Liu

liujc@math.ntnu.edu.tw

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Chapter II

THE STRUCTURE OF GROUPS

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Hence we have

Order *Distinct Groups*

1

2

3

4

5

6

7

8

9

10

11

12

Order *Distinct Groups*

13

14

15

Order *Distinct Groups*

1 $\langle e \rangle$

2

3

4

5

6

7

8

9

10

11

12

Order *Distinct Groups*

13

14

15

Order *Distinct Groups*

1 $\langle e \rangle$

2 \mathbb{Z}_2

3

4

5

6

7

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4

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13

14

15

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3 \mathbb{Z}_3

4

5 \mathbb{Z}_5

6

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8

9

10

11

12

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13

14

15

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2 \mathbb{Z}_2

3 \mathbb{Z}_3

4

5 \mathbb{Z}_5

6

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8

9

10

11 \mathbb{Z}_{11}

12

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13

14

15

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2 \mathbb{Z}_2

3 \mathbb{Z}_3

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5 \mathbb{Z}_5

6

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8

9

10

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14

15

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Order *Distinct Groups*

1 $\langle e \rangle$

2 \mathbb{Z}_2

3 \mathbb{Z}_3

4 $\mathbb{Z}_2 \oplus \mathbb{Z}_2, \mathbb{Z}_4$

5 \mathbb{Z}_5

6

7 \mathbb{Z}_7

8

9 $\mathbb{Z}_3 \oplus \mathbb{Z}_3, \mathbb{Z}_9$

10

11 \mathbb{Z}_{11}

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Order $2p$ with Odd Prime p

Review: Dihedral Groups

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- (ii) $ba = a^{-1}b$, i.e., $bab^{-1} = a^{-1}$.

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Definition. The **dihedral group of degree n** , denoted by D_n , is the subgroup of S_n generated by $a = (1, 2, 3, \dots, n)$ and

$$b = \prod_{2 \leq i \leq \lfloor \frac{n-1}{2} \rfloor} (i, n+2-i)$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & i & \cdots & n-1 & n \\ 1 & n & n-1 & n-2 & \cdots & n+2-i & \cdots & 3 & 2 \end{pmatrix}.$$

Theorem (6.13). For each $n \geq 3$, the dihedral group D_n is a group of order $2n$ whose generators a and b satisfy

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 - If $r = 1$, i.e., $bab^{-1} = a$, then $ba = ab$ and so $G = \langle a, b \rangle$ is abelian. Hence P and Q are both normal subgroup of G with $P \cap Q = \{e\}$ and $G = PQ$, so $G \cong P \times Q \cong \mathbb{Z}_p \times \mathbb{Z}_2 \cong \mathbb{Z}_{2p}$.
 - If $r = p - 1$, i.e., $bab^{-1} = a^{p-1} = a^{-1}$. Since $G = \langle a, b \rangle$, $G \cong D_p$.

Therefore, we have

Order *Distinct Groups*

1 $\langle e \rangle$

2 \mathbb{Z}_2

3 \mathbb{Z}_3

4 $\mathbb{Z}_2 \oplus \mathbb{Z}_2, \mathbb{Z}_4$

5 \mathbb{Z}_5

6

7 \mathbb{Z}_7

8

9 $\mathbb{Z}_3 \oplus \mathbb{Z}_3, \mathbb{Z}_9$

10

11 \mathbb{Z}_{11}

12

Order *Distinct Groups*

13 \mathbb{Z}_{13}

14

15 \mathbb{Z}_{15}

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6 $\mathbb{Z}_6, D_3 = S_3$

7 \mathbb{Z}_7

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$|Q_8| \leq 8$. Furthermore, it is clear that $B \notin \langle A \rangle$, so

$|Q_8| > |A| = 4$. Moreover, since $4 = |A| \mid |Q_8|$, so $|Q_8| = 8$, i.e.,

Q_8 is a group of order 8.

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Since $[G : \langle a \rangle] = 2$, $\langle a \rangle \triangleleft G$ and $G/\langle a \rangle$ is a group of order 2.

Take $b \in G \setminus \langle a \rangle$. Then $G = \langle a, b \rangle$ and $b^2 \in \langle a \rangle$, i.e., $b^2 = e, a, a^2$, or a^3 .

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Hence $b^2 = e$ or $b^2 = a^2$.

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- If $b^2 = e$, then $G \cong D_4$.
- If $b^2 = a^2$, then $G \cong Q_8$.

Order *Distinct Groups*

1 $\langle e \rangle$

2 \mathbb{Z}_2

3 \mathbb{Z}_3

4 $\mathbb{Z}_2 \oplus \mathbb{Z}_2, \mathbb{Z}_4$

5 \mathbb{Z}_5

6 $\mathbb{Z}_6, D_3 = S_3$

7 \mathbb{Z}_7

8

9 $\mathbb{Z}_3 \oplus \mathbb{Z}_3, \mathbb{Z}_9$

10 \mathbb{Z}_{10}, D_5

11 \mathbb{Z}_{11}

12

Order *Distinct Groups*

13 \mathbb{Z}_{13}

14 \mathbb{Z}_{14}, D_7

15 \mathbb{Z}_{15}

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1 $\langle e \rangle$

2 \mathbb{Z}_2

3 \mathbb{Z}_3

4 $\mathbb{Z}_2 \oplus \mathbb{Z}_2, \mathbb{Z}_4$

5 \mathbb{Z}_5

6 $\mathbb{Z}_6, D_3 = S_3$

7 \mathbb{Z}_7

8 $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2,$

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10 \mathbb{Z}_{10}, D_5

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7 \mathbb{Z}_7

8 $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2, \mathbb{Z}_2 \oplus \mathbb{Z}_4,$

9 $\mathbb{Z}_3 \oplus \mathbb{Z}_3, \mathbb{Z}_9$

10 \mathbb{Z}_{10}, D_5

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8 $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2, \mathbb{Z}_2 \oplus \mathbb{Z}_4, \mathbb{Z}_8,$

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6 $\mathbb{Z}_6, D_3 = S_3$

7 \mathbb{Z}_7

8 $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2, \mathbb{Z}_2 \oplus \mathbb{Z}_4, \mathbb{Z}_8, Q_8,$

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Groups of Order 12

Let G be a group of order 12.

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Groups of Order 12

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- On the other hand, we already know two nonabelian groups of order 12,

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Hence we have $|a| = 6$, $|b| = 4$, and $\langle a \rangle \cap \langle b \rangle = \{e, a^3 = b^2\}$.

Therefore, if $a^\ell = b^k$ with $\ell, k \in \mathbb{Z}$, then $3 \mid \ell$ and $2 \mid k$.

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$$bab^{-1} = (\tau\sigma\tau^{-1}, \bar{1} + \bar{2} - \bar{1}) = (\sigma^2, \bar{2}) = a^5 = a^{-1}.$$

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- $T = \langle a, b \rangle = \{a^i b^j \mid 0 \leq i \leq 5 \text{ and } 0 \leq j \leq 1\}$.

If $a^i b^j = a^s b^t$ with $0 \leq i, s \leq 5$ and $0 \leq j, t \leq 1$, then $a^{i-s} = b^{t-j}$ and so $2 \mid t - j$. Since $-1 \leq t - j \leq 1$, $t - j = 0$, i.e., $t = j$, and this implies $i = s$. Therefore, $|T| = 12$, i.e., T is a group of order 12.

Next, we show that if G is a group generated by $\alpha, \beta \in G$ satisfying $|\alpha| = 6, \alpha^3 = \beta^2$, and $\beta\alpha = \alpha^{-1}\beta$,

A Group of Order 12

Consider the group $S_3 \times \mathbb{Z}_4$ and let $a = (\sigma, \bar{2})$ and $b = (\tau, \bar{1})$ in $S_3 \times \mathbb{Z}_4$, where $\sigma = (1, 2, 3)$ and $\tau = (1, 2)$ in S_3 . Let T be the subgroup of $S_3 \times \mathbb{Z}_4$ generated by a, b , i.e., $T = \langle a, b \rangle$. We have

- $|a| = 6, |b| = 4, a^3 = b^2$, and $ba = a^{-1}b$;
- if $a^\ell = b^k$ with $\ell, k \in \mathbb{Z}$, then $3 \mid \ell$ and $2 \mid k$;
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Next, we show that if G is a group generated by $\alpha, \beta \in G$ satisfying $|\alpha| = 6, \alpha^3 = \beta^2$, and $\beta\alpha = \alpha^{-1}\beta$, then G is isomorphic to T .

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Moreover, since $\beta^2 = \alpha^3 \neq e$ and $\beta^4 = \alpha^6 = e$, $|\beta| = 4$. Recall that $T = \langle a, b \rangle = \{ a^i b^j \mid 0 \leq i \leq 5 \text{ and } 0 \leq j \leq 1 \}$, and these elements are distinct. Hence, we can define a map $\phi : T \rightarrow G$ by $a^i b^j \mapsto \alpha^i \beta^j$. It is also clear that ϕ is onto. Moreover, since $ba = a^{-1}b$ and $\beta\alpha = \alpha^{-1}\beta$, we see that ϕ is indeed a homomorphism. Finally, let $a^i b^j \in \text{Ker } \phi$. Then $\alpha^i \beta^j = e$ in G .

If $j = 1$, then

$$\alpha^i \beta = e \implies \beta = \alpha^{-i} \in \langle \alpha \rangle \implies |\beta| \mid |\langle \alpha \rangle| \implies 4 \mid 6,$$

but 4 does not divide 6. Hence $j = 0$, i.e., $\alpha^i = e$, and this implies $i = 0$, i.e., $a^i b^j = e$ in T . Thus, ϕ is one-to-one

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Next, we will show that up to isomorphism, there are exactly three distinct nonabelian groups of order 12.