Modern Algebra I

Lecture 11

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Today, we will cover the following two sections

Section II.4: The Action of a Group on a Set

Section II.5: The Sylow Theorems

Chapter II

The Structure of Groups

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Example. Let $I_n = \{1, 2, \ldots, n\}$ and let S_n be the symmetric group on I_n. Then the function $S_n \times I_n \to I_n$ given by $(\sigma, x) \mapsto \sigma(x)$

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  We describe this action as H acts on S by translation.

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is an equivalence relation.

(ii) For each $x \in S$,

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is a subgroup of G.

Proof. For (ii), first note that $ex = x$, so $e \in G_x$. Secondly, let $g_1, g_2 \in G_x$. Then $g_1 x = x = g_2 x$. Since $(g_1 g_2)x = g_1(g_2 x)$ $= g_1 x = x$, $g_1 g_2 \in G_x$. Finally, let $g \in G_x$. Then $gx = x$. Since $g^{-1}x = g^{-1}(gx) = (g^{-1}g)x = ex = x$, $g^{-1} \in G_x$. Therefore, G_x is a subgroup of G.

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This equation is called the class equation of the finite group G.

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Corollary (4.6, Cayley)**.** If G is a group,

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Proof. Let G act on itself by left translation. By the proof of Theorem (4.5), $\Phi : G \to A(G)$ with $g \mapsto \tau_g$ is a homomorphism.

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Proof. Let G act on itself by left translation. By the proof of Theorem (4.5), $\Phi : G \to A(G)$ with $g \mapsto \tau_g$ is a homomorphism. Moveover,

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Proof. Apply Proposition (4.8) to H. Since $[G : H] = p$, $A(S) \simeq S_p$. Let $\Phi : G \to A(S)$ be the induced homomorphism and let $K = \operatorname{Ker} \Phi$. By Proposition (4.8), $K \leq H$. On the other hand, by the First Isomorphism Theorem, $G/K \simeq \Phi(G) \leq A(S)$. Hence, by Lagrange's Theorem,

$$[G : K] = |G/K| \Big| |A(S)| = |S_p| = p!$$

Furthermore, $[G : K] \Big| |G|$ and p is the smallest prime dividing $|G|$, so $[G : K] = 1$ or p. On the other hand, $K \leq H \implies [G : K] \geq [G : H] = p$, so $[G : K] = p = [G : H]$

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Exercise for Section II.4

1, 2, 3, 7, 8, 9, 14.