Modern Algebra I

Lecture 10

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2009, Fall

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Section II.2: Finitely Generated Abelian Groups

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Conversely, if $u \in \sum_{i \in I} (G_i)_t$, i.e., $a_i \in (G_i)_t$ for all $i \in I$, then for each $i \in I$, $\exists n_i \in \mathbb{N}$ such that $n_i a_i = 0$. Take $n = n_{i_1} n_{i_2} \cdots n_{i_k}$. Then $na_i = 0$ for all $i \in I$. This implies $nu = 0$

Lemma 1

Let $\{G_i \mid i \in I\}$ be a family of abelian groups and let $G = \sum_{i \in I} G_i$. Let $m \in \mathbb{Z}$ and let p be a prime number. Then

- $mG = \sum_{i \in I} mG_i$
- $G[m] = \sum_{i \in I} G_i[m]$
- $G(p) = \sum_{i \in I} G_i(p)$
- $G_t = \sum_{i \in I} (G_i)_t$.

Proof. Let $u = (a_i)_{i \in I} \in G = \sum_{i \in I} G_i$, and let $I_{\neq 0} = \{i_1, \ldots, i_k\}$.

If $u \in G_t$, then $nu = 0$ for some $n \in \mathbb{N}$. Then $(na_i)_{i \in I} = 0$, i.e., $na_i = 0$ for all $i \in I$. Hence $a_i \in (G_i)_t$ for all $i \in I$ and this implies $u = (a_i)_{i \in I} \in \sum_{i \in I} (G_i)_t$.

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Lemma (2.5). Let G be an abelian group, $m \in \mathbb{Z}$, p a prime number. Then each of the following is a subgroup of G:

(i) $mG = \{mu \mid u \in G\}$;

(ii) $G[m] = \{u \in G \mid mu = 0\}$;

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- $\mathbb{Z}_{p^n}[p] \cong \mathbb{Z}_p$ for all $n \in \mathbb{N}$.
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We are ready to state and prove the last two statements.

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Proof. Theorem (2.1) tells us that G is a finite direct sum of cyclic groups.

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- if $p_i = p$ and $n_i \leq m$, then $(p^m \mathbb{Z}_{p_i^{n_i}})[p] = 0$;
- if $p_i = p$ and $n_i > m$, then $(p^m \mathbb{Z}_{p_i^{n_i}})[p] \cong \mathbb{Z}_{p^{n_i - m}}[p] \cong \mathbb{Z}_p$;
- $(p^m F)[p] = 0$.

Hence if r is the number of i such that $p_i = p$ and $n_i > m$, then $|(p^m G)[p]| = p^r$. Since $(p^m G)[p]$ does not depend on the particular decomposition,

Theorem (2.6)**.** Let G be a finitely generated abelian group.

(iii) Either G is free abelian or there is a list of positive integers $p_1^{n_1}, \ldots, p_k^{n_k}$ which is unique except for the order of its members, such that $p_1, \ldots, p_k$ are (not necessarily distinct) primes, $n_1, \ldots, n_k$ are (not necessarily distinct) positive integers and $G \cong \mathbb{Z}_{p_1^{n_1}} \oplus \cdots \oplus \mathbb{Z}_{p_k^{n_k}} \oplus F$ with F free abelian.

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Lemma (2.3)**.** If m is a positive integer and $m = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$ with $p_1, \ldots, p_k$ distinct primes and $n_i \in \mathbb{N}$, then $\mathbb{Z}_m \cong \mathbb{Z}_{p_1^{n_1}} \oplus \cdots \oplus \mathbb{Z}_{p_k^{n_k}}$.

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Remark. If $G \cong \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_\ell} \oplus F$ with F free abelian, we know that we can apply Lemma (2.3) to decompose G further

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Remark. If $G \cong \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_\ell} \oplus F$ with F free abelian, we know that we can apply Lemma (2.3) to decompose G further and obtain a decomposition of G in the form $G \cong \mathbb{Z}_{p_1^{n_1}} \oplus \cdots \oplus \mathbb{Z}_{p_k^{n_k}} \oplus F$.

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Conversely, if $G \cong \mathbb{Z}_{p_1^{n_1}} \oplus \cdots \oplus \mathbb{Z}_{p_k^{n_k}} \oplus F$ with F free abelian, we can also apply Lemma (2.3) to "combine" those summands with distinct primes p_i together.

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In other words, there is a one-to-one correspondence between the decompositions $G \cong \mathbb{Z}_{p_1^{n_1}} \oplus \cdots \oplus \mathbb{Z}_{p_k^{n_k}} \oplus F$ in (ii)

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Remark. If $G \cong \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_\ell} \oplus F$ with F free abelian, we know that we can apply Lemma (2.3) to decompose G further and obtain a decomposition of G in the form $G \cong \mathbb{Z}_{p_1^{n_1}} \oplus \cdots \oplus \mathbb{Z}_{p_k^{n_k}} \oplus F$.
Conversely, if $G \cong \mathbb{Z}_{p_1^{n_1}} \oplus \cdots \oplus \mathbb{Z}_{p_k^{n_k}} \oplus F$ with F free abelian, we can also apply Lemma (2.3) to "combine" those summands with distinct primes p_i together. In fact, if we further require that $m_1 \mid m_2 \mid \cdots \mid m_\ell$, there is only a unique corresponding $G \cong \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_\ell} \oplus F$.

In other words, there is a one-to-one correspondence between the decompositions $G \cong \mathbb{Z}_{p_1^{n_1}} \oplus \cdots \oplus \mathbb{Z}_{p_k^{n_k}} \oplus F$ in (ii) and the decomposition $G \cong \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_\ell} \oplus F$ in (iii).

Example

Suppose $G \cong \mathbb{Z}_5 \oplus \mathbb{Z}_{15} \oplus \mathbb{Z}_{25} \oplus \mathbb{Z}_{36} \oplus \mathbb{Z}_{56}$.

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which is the decomposition in (iii).

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Next, we see how to obtain the decomposition in (ii).

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which is the decomposition in (iii).
Next, we see how to obtain the decomposition in (ii).
Collect the powers of the same prime together:

$$G \quad \cong$$

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which is the decomposition in (iii).

Next, we see how to obtain the decomposition in (ii).

Collect the powers of the same prime together: powers of 2

$$G \cong (\mathbb{Z}_{2^3} \oplus \mathbb{Z}_{2^2})$$

Example

Suppose $G \cong \mathbb{Z}_5 \oplus \mathbb{Z}_{15} \oplus \mathbb{Z}_{25} \oplus \mathbb{Z}_{36} \oplus \mathbb{Z}_{56}$.

First, we apply Lemma (2.3) and obtain

$$G \cong \mathbb{Z}_5 \oplus (\mathbb{Z}_3 \oplus \mathbb{Z}_5) \oplus \mathbb{Z}_{5^2} \oplus (\mathbb{Z}_{2^2} \oplus \mathbb{Z}_{3^2}) \oplus (\mathbb{Z}_{2^3} \oplus \mathbb{Z}_7),$$

which is the decomposition in (iii).

Next, we see how to obtain the decomposition in (ii).

Collect the powers of the same prime together: powers of 3

$$G \cong (\mathbb{Z}_{2^3} \oplus \mathbb{Z}_{2^2}) \oplus (\mathbb{Z}_{3^2} \oplus \mathbb{Z}_3)$$

Example

Suppose $G \cong \mathbb{Z}_5 \oplus \mathbb{Z}_{15} \oplus \mathbb{Z}_{25} \oplus \mathbb{Z}_{36} \oplus \mathbb{Z}_{56}$.
First, we apply Lemma (2.3) and obtain

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which is the decomposition in (iii).
Next, we see how to obtain the decomposition in (ii).
Collect the powers of the same prime together: powers of 5

$$G \cong (\mathbb{Z}_{2^3} \oplus \mathbb{Z}_{2^2}) \oplus (\mathbb{Z}_{3^2} \oplus \mathbb{Z}_3) \oplus (\mathbb{Z}_{5^2} \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5)$$

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Suppose $G \cong \mathbb{Z}_5 \oplus \mathbb{Z}_{15} \oplus \mathbb{Z}_{25} \oplus \mathbb{Z}_{36} \oplus \mathbb{Z}_{56}$.

First, we apply Lemma (2.3) and obtain

$$G \cong \mathbb{Z}_5 \oplus (\mathbb{Z}_3 \oplus \mathbb{Z}_5) \oplus \mathbb{Z}_{5^2} \oplus (\mathbb{Z}_{2^2} \oplus \mathbb{Z}_{3^2}) \oplus (\mathbb{Z}_{2^3} \oplus \mathbb{Z}_7),$$

which is the decomposition in (iii).

Next, we see how to obtain the decomposition in (ii).

Collect the powers of the same prime together: powers of 7

$$G \quad \cong \quad (\mathbb{Z}_{2^3} \oplus \mathbb{Z}_{2^2}) \oplus (\mathbb{Z}_{3^2} \oplus \mathbb{Z}_3) \oplus (\mathbb{Z}_{5^2} \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5) \oplus \mathbb{Z}_7$$

Example

Suppose $G \cong \mathbb{Z}_5 \oplus \mathbb{Z}_{15} \oplus \mathbb{Z}_{25} \oplus \mathbb{Z}_{36} \oplus \mathbb{Z}_{56}$.

First, we apply Lemma (2.3) and obtain

$$G \cong \mathbb{Z}_5 \oplus (\mathbb{Z}_3 \oplus \mathbb{Z}_5) \oplus \mathbb{Z}_{5^2} \oplus (\mathbb{Z}_{2^2} \oplus \mathbb{Z}_{3^2}) \oplus (\mathbb{Z}_{2^3} \oplus \mathbb{Z}_7),$$

which is the decomposition in (iii).

Next, we see how to obtain the decomposition in (ii).

Collect the highest powers for each prime

$$\begin{aligned} G &\cong (\mathbb{Z}_{2^3} \oplus \mathbb{Z}_{2^2}) \oplus (\mathbb{Z}_{3^2} \oplus \mathbb{Z}_3) \oplus (\mathbb{Z}_{5^2} \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5) \oplus \mathbb{Z}_7 \\ &\cong (\mathbb{Z}_{2^3} \oplus \mathbb{Z}_{3^2} \oplus \mathbb{Z}_{5^2} \oplus \mathbb{Z}_7) \end{aligned}$$

Example

Suppose $G \cong \mathbb{Z}_5 \oplus \mathbb{Z}_{15} \oplus \mathbb{Z}_{25} \oplus \mathbb{Z}_{36} \oplus \mathbb{Z}_{56}$.

First, we apply Lemma (2.3) and obtain

$$G \cong \mathbb{Z}_5 \oplus (\mathbb{Z}_3 \oplus \mathbb{Z}_5) \oplus \mathbb{Z}_{5^2} \oplus (\mathbb{Z}_{2^2} \oplus \mathbb{Z}_{3^2}) \oplus (\mathbb{Z}_{2^3} \oplus \mathbb{Z}_7),$$

which is the decomposition in (iii).

Next, we see how to obtain the decomposition in (ii).

Collect the second highest powers for each prime

$$\begin{aligned} G &\cong (\mathbb{Z}_{2^3} \oplus \mathbb{Z}_{2^2}) \oplus (\mathbb{Z}_{3^2} \oplus \mathbb{Z}_3) \oplus (\mathbb{Z}_{5^2} \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5) \oplus \mathbb{Z}_7 \\ &\cong (\mathbb{Z}_{2^3} \oplus \mathbb{Z}_{3^2} \oplus \mathbb{Z}_{5^2} \oplus \mathbb{Z}_7) \oplus (\mathbb{Z}_{2^2} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5) \end{aligned}$$

Example

Suppose $G \cong \mathbb{Z}_5 \oplus \mathbb{Z}_{15} \oplus \mathbb{Z}_{25} \oplus \mathbb{Z}_{36} \oplus \mathbb{Z}_{56}$.
First, we apply Lemma (2.3) and obtain

$$G \cong \mathbb{Z}_5 \oplus (\mathbb{Z}_3 \oplus \mathbb{Z}_5) \oplus \mathbb{Z}_{5^2} \oplus (\mathbb{Z}_{2^2} \oplus \mathbb{Z}_{3^2}) \oplus (\mathbb{Z}_{2^3} \oplus \mathbb{Z}_7),$$

which is the decomposition in (iii).
Next, we see how to obtain the decomposition in (ii).
Collect the third highest powers for each prime

$$\begin{aligned} G &\cong (\mathbb{Z}_{2^3} \oplus \mathbb{Z}_{2^2}) \oplus (\mathbb{Z}_{3^2} \oplus \mathbb{Z}_3) \oplus (\mathbb{Z}_{5^2} \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5) \oplus \mathbb{Z}_7 \\ &\cong (\mathbb{Z}_{2^3} \oplus \mathbb{Z}_{3^2} \oplus \mathbb{Z}_{5^2} \oplus \mathbb{Z}_7) \oplus (\mathbb{Z}_{2^2} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5) \oplus \mathbb{Z}_5 \end{aligned}$$

Example

Suppose $G \cong \mathbb{Z}_5 \oplus \mathbb{Z}_{15} \oplus \mathbb{Z}_{25} \oplus \mathbb{Z}_{36} \oplus \mathbb{Z}_{56}$.

First, we apply Lemma (2.3) and obtain

$$G \cong \mathbb{Z}_5 \oplus (\mathbb{Z}_3 \oplus \mathbb{Z}_5) \oplus \mathbb{Z}_{5^2} \oplus (\mathbb{Z}_{2^2} \oplus \mathbb{Z}_{3^2}) \oplus (\mathbb{Z}_{2^3} \oplus \mathbb{Z}_7),$$

which is the decomposition in (iii).

Next, we see how to obtain the decomposition in (ii).

Apply Lemma (3.2)

$$\begin{aligned} G &\cong (\mathbb{Z}_{2^3} \oplus \mathbb{Z}_{2^2}) \oplus (\mathbb{Z}_{3^2} \oplus \mathbb{Z}_3) \oplus (\mathbb{Z}_{5^2} \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5) \oplus \mathbb{Z}_7 \\ &\cong (\mathbb{Z}_{2^3} \oplus \mathbb{Z}_{3^2} \oplus \mathbb{Z}_{5^2} \oplus \mathbb{Z}_7) \oplus (\mathbb{Z}_{2^2} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5) \oplus \mathbb{Z}_5 \\ &\cong \mathbb{Z}_{2^3 . 3^2 . 5^2 . 7} \end{aligned}$$

Example

Suppose $G \cong \mathbb{Z}_5 \oplus \mathbb{Z}_{15} \oplus \mathbb{Z}_{25} \oplus \mathbb{Z}_{36} \oplus \mathbb{Z}_{56}$.
First, we apply Lemma (2.3) and obtain

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which is the decomposition in (iii).
Next, we see how to obtain the decomposition in (ii).
Apply Lemma (3.2)

$$\begin{aligned} G &\cong (\mathbb{Z}_{2^3} \oplus \mathbb{Z}_{2^2}) \oplus (\mathbb{Z}_{3^2} \oplus \mathbb{Z}_3) \oplus (\mathbb{Z}_{5^2} \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5) \oplus \mathbb{Z}_7 \\ &\cong (\mathbb{Z}_{2^3} \oplus \mathbb{Z}_{3^2} \oplus \mathbb{Z}_{5^2} \oplus \mathbb{Z}_7) \oplus (\mathbb{Z}_{2^2} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5) \oplus \mathbb{Z}_5 \\ &\cong \mathbb{Z}_{2^3 \cdot 3^2 \cdot 5^2 \cdot 7} \oplus \mathbb{Z}_{2^2 \cdot 3 \cdot 5} \end{aligned}$$

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Suppose $G \cong \mathbb{Z}_5 \oplus \mathbb{Z}_{15} \oplus \mathbb{Z}_{25} \oplus \mathbb{Z}_{36} \oplus \mathbb{Z}_{56}$.

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which is the decomposition in (iii).

Next, we see how to obtain the decomposition in (ii).

Apply Lemma (3.2)

$$\begin{aligned} G &\cong (\mathbb{Z}_{2^3} \oplus \mathbb{Z}_{2^2}) \oplus (\mathbb{Z}_{3^2} \oplus \mathbb{Z}_3) \oplus (\mathbb{Z}_{5^2} \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5) \oplus \mathbb{Z}_7 \\ &\cong (\mathbb{Z}_{2^3} \oplus \mathbb{Z}_{3^2} \oplus \mathbb{Z}_{5^2} \oplus \mathbb{Z}_7) \oplus (\mathbb{Z}_{2^2} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5) \oplus \mathbb{Z}_5 \\ &\cong \mathbb{Z}_{2^3 \cdot 3^2 \cdot 5^2 \cdot 7} \oplus \mathbb{Z}_{2^2 \cdot 3 \cdot 5} \oplus \mathbb{Z}_5 \end{aligned}$$

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First, we apply Lemma (2.3) and obtain

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Use your calculator

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Reversing the order of the summands to fit Theorem (2.6)

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Hence, $G \cong \mathbb{Z}_5 \oplus \mathbb{Z}_{60} \oplus \mathbb{Z}_{12600}$ is the decomposition in (ii).

Theorem (2.6)**.** Let G be a finitely generated abelian group.

(ii) Either G is free abelian or there is a unique list of (not necessarily distinct) positive integers $m_1, \ldots, m_\ell$ such that $m_1 > 1$, $m_1 \mid m_2 \mid \cdots \mid m_\ell$ and $G \cong \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_\ell} \oplus F$ with F free abelian.

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Hence the invariant factors of G are 5, 60, 12600; the elementary divisors of G are 2^2, 2^3, 3, 3^2, 5, 5, 5^2, 7.

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with $m_1 > 1$, $m_1 \mid m_2 \mid \cdots \mid m_t$, $p_1, \ldots, p_\ell$ primes (not necessarily distinct), and $n_1 \ldots, n_\ell \in \mathbb{N}$.

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$$\begin{aligned} G &\cong \mathbb{Z}_{m_1} \oplus \mathbb{Z}_{m_2} \oplus \cdots \oplus \mathbb{Z}_{m_t} \oplus \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{s \text{ summands}} \\ &\cong \mathbb{Z}_{p_1^{n_1}} \oplus \mathbb{Z}_{p_2^{n_2}} \oplus \cdots \oplus \mathbb{Z}_{p_\ell^{n_\ell}} \oplus \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{s \text{ summands}} \end{aligned}$$

with $m_1 > 1$, $m_1 \mid m_2 \mid \cdots \mid m_t$, $p_1, \ldots, p_\ell$ primes (not necessarily distinct), and $n_1 \ldots, n_\ell \in \mathbb{N}$.

- $G/G_t \cong \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{s \text{ summands}}$, which is a free abelian group of rank s.

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Then G/G_t has rank s and H/H_t has rank ℓ.

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Exercise for Section II.2

1, 5, 6, 11, 12, 13.