Modern Algebra I

Lecture 1

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Chapter I: Groups

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Section I.1: Semigroups, Monoids, and Groups

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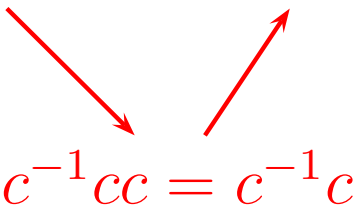

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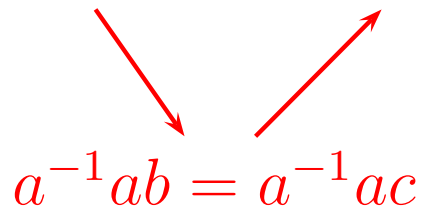

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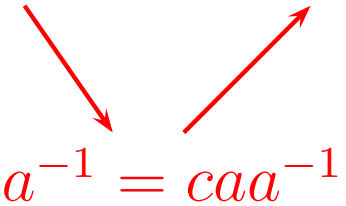

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Proof. We need to show (1) $ae = a, \forall a \in G$, and (2) $aa^{-1} = e, \forall a \in G$. Note that

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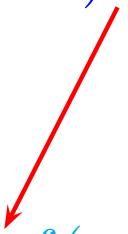

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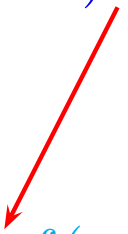

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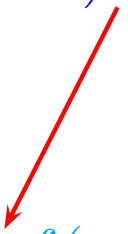

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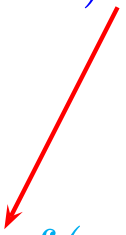

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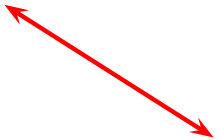

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"$\Rightarrow$": $a, b \in H \implies a, b^{-1} \in H \implies ab^{-1} \in H$.

"$\Leftarrow$": Because $H \neq \varnothing$, $\exists a \in H$. Then $e = aa^{-1} \in H$ and this proves (2). Next, $a \in H \implies e, a \in H \implies a^{-1} = ea^{-1} \in H$. This shows (3). Finally,

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Hence, by Theorem (2.5), H is a subgroup of G.

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Proof. This is because $\mathrm{Ker}\ f = f^{-1}(\{e\})$.

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Corollary (2.6). If G is a group and if $\{H_i \mid i \in I\}$ is a family of subgroups of G, then $\bigcap_{i \in I} H_i$ is a subgroup of G.

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Question. What is $\langle \varnothing \rangle$? **Answer.** $\langle \varnothing \rangle = \{e\}$.

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Therefore, $\mathbb{Z}_m \cong \operatorname{Im}\varphi = \langle a \rangle$ and $|\langle a \rangle| = m$.

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Proof. Because $|a| = \infty$,

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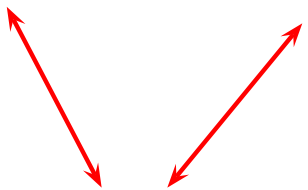

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Proof. "$\Leftarrow$" is clear. For "$\Rightarrow$", $b \in G = \langle a \rangle$, so $b = a^n$ for some $n \in \mathbb{Z}$. Conversely, $a \in G = \langle b \rangle$,

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Proof. "$\Leftarrow$" is clear. For "$\Rightarrow$", $b \in G = \langle a \rangle$, so $b = a^n$ for some $n \in \mathbb{Z}$. Conversely, $a \in G = \langle b \rangle$, so $a = b^k$ for some $k \in \mathbb{Z}$.

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